

An analytical phase-space model for tidal caustics

Robyn E. Sanderson, Amina Helmi¹

¹*Kapteyn Astronomical Institute, P.O. Box 800, 9700 AV Groningen, The Netherlands*

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ABSTRACT

The class of tidal features around galaxies known variously as “shells” or “umbrellas” comprises debris that has arisen from high-mass-ratio mergers with low impact parameter; the nearly radial orbits of the debris give rise to a unique morphology, a universal density profile, and a tight correlation between positions and velocities of the material. As such they are accessible to analytical treatment, and can provide a relatively clean system for probing the gravitational potential of the host galaxy. In this work we present a simple analytical model that describes the density profile, phase-space distribution, and geometry of a shell, and whose parameters are directly related to physical characteristics of the interacting galaxies. The model makes three assumptions: that their orbit is radial, that the potential of the host is spherical at the shell radii, and that the satellite galaxy had a Maxwellian velocity distribution. We quantify the error introduced by the first two assumptions and show that selecting shells by their appearance on the sky is a sufficient basis to assume that these simplifications are valid. We further demonstrate that (1) given only an image of a shell, the radial gravitational force at the shell edge and the phase-space density of the satellite are jointly constrained, (2) that combining the image with measurements of either point line-of-sight velocities or integrated spectra will yield an independent estimate of the gravitational force at a shell, and (3) that an independent measurement of this force is obtained for each shell observed around a given galaxy, potentially enabling a determination of the galactic mass distribution.

Key words:

1 INTRODUCTION

Recently, large-scale sky surveys and deep follow-up images have been used to discover a wealth of tidal debris around our Galaxy and others nearby (e.g., Ibata et al. 2001; McConnachie et al. 2009; Trujillo et al. 2009; Martínez-Delgado et al. 2010; Radburn-Smith et al. 2011). This tidal debris comes in many shapes and sizes, from the huge tidal arms created by interacting, roughly equal-mass galaxies to the fainter features observed around nearby galaxies. These smaller-scale features are thought to come from interactions of the large galaxy with much smaller satellites, which are called minor mergers. Evidence that minor mergers occur in nature is an important link to our cosmological history, since cosmological simulations of dark matter indicate that about half the mass in the Milky Way’s outer regions was accreted in this way (for example, Maciejewski et al. 2011; Wang et al. 2011). Minor mergers are also useful as a way to constrain the shape and mass of the large galaxy, since tidally stripped material from the smaller satellite galaxy behaves as test particles in the relatively undisturbed potential of the larger host galaxy (Ibata et al. 2001; Helmi 2004; Johnston et al. 2005; Eyre & Binney 2009; Law &

Majewski 2010). The remnants of these mergers give us a way to measure the characteristics of the dark components of galaxies, which are predicted with great accuracy by cosmological models and simulations.

Some minor mergers create patterns of tidal debris that look like shells or umbrellas. The first such debris was identified around elliptical galaxies by Malin & Carter (1983); these galaxies were called “shell galaxies” because of these distinctive features. Hernquist & Quinn (1987, 1988, 1989) showed that the shells were probably created by a minor merger on a nearly radial orbit. This explains the alternate spacing of the shells on either side of the host galaxy since they are formed as material from the satellite piles up approximately at turning points: the satellite initially had a distribution of energies that is reflected in the different radii of the shells. More recently, similar features have been discovered around nearby disk galaxies (Ibata et al. 2001; McConnachie et al. 2009; Martínez-Delgado et al. 2010). The vast improvements in imaging since shell galaxies were first identified, and the relative proximity of these objects, have revealed more of their structure than had previously been observed. In some cases (Fardal et al. 2012; Romanowsky

et al. 2012), the objects are even close or bright enough that velocity information could be obtained.

Shells from nearly radial mergers are particularly special because there is a direct correlation between the kinematic properties of debris and its location relative to the host galaxy. Thanks to the near-symmetry of the encounter, the system can be considered in a two-dimensional projection (r, v_r) of the full six-dimensional phase space without much loss of information. In this two-dimensional projection, the initially cold satellite material, once unbound, forms a thin stream that winds through phase space, so that for any spatial location r there are a small, finite number of streams with different characteristic v_r . Merrifield & Kuijken (1998) pointed out that this correlation could be used to estimate the mass of the galaxy hosting a shell, if the line-of-sight velocity of the material could be measured at different points in the shell.

The dynamics governing the formation and shape of this stream are closely related to earlier work on spherically symmetric secondary infall of matter accreting onto dark matter halos. As shown in Fillmore & Goldreich (1984) and Bertschinger (1985) radial accretion of gravitating, cold, collisionless matter forms a series of infinite-density peaks at successive radii, known as caustics. Mohayaee & Shandarin (2006) further showed that for warm matter with a finite velocity dispersion, the peaks take on a finite width and height, but although they are no longer caustics in the mathematically rigorous sense they retain many of the same properties. This work was motivated by cosmological simulations and so considered the radial collapse of spherically distributed matter, but Helmi & White (1999) showed that caustics are also produced when a small, initially self-bound satellite falls into the (assumedly) static potential of a larger host galaxy. In fact, caustics are a universal product of dynamical systems with turning points, regardless of the type of symmetry (Tremaine 1999; Hogan 2001). This work has been recently confirmed with state-of-the-art computer simulations using realistic galactic potentials by Vogelsberger et al. (2008), and systems resembling shell galaxies are produced in cosmological simulations with semi-analytic stellar components (Cooper et al. 2011). Sanderson & Bertschinger (2010) demonstrated the connection between the shape of the density profile of the caustic and the physical characteristics of the interacting galaxies. For these reasons, in this work we will refer to shells as “tidal caustics”, a term which emphasizes their high degree of symmetry and the correlation of positions and velocities.

Recently, Ebrov et al. (2012) have discussed the possibility of using integrated-light spectra to measure the gravitational potential in shell galaxies by obtaining the line-of-sight velocity profiles of the shell debris. Their results are similar to some of those given in this work, with a few important differences. First, they consider only spherical shells from satellites on perfectly radial orbits, not the effect of relaxing these assumptions to include angular momentum, potential flattening, or projection effects. In this work we explore how all three of these things affect the line-of sight velocity distribution. Furthermore, the line-of-sight velocity profiles derived by Ebrov et al. do not account for the nonzero thickness of the shell. This causes the peaks of the profile in their model to be thinner than and slightly offset from the peaks in their simulated shells; we show in this

work that including the shell thickness solves both problems. Finally, they do not discuss how the surface-brightness profile of the shell is related to the kinematic profile, whereas we show that the two can be used together to simplify the fitting process by constraining the shell’s geometry independently prior to modelling its kinematics.

In this paper we present a simple analytical model for the density and phase space distributions of tidal caustics and relate the parameters of the model to characteristics of the interacting galaxies (Sections 3 and 4), specifically the radial component of the gravitational force g_s produced by the host galaxy at the radius of a tidal caustic and the initial phase space density of the satellite galaxy. We discuss how the geometry of the caustic can be modeled in order to calculate projections of the two analytic distributions (Section 5) and then present three examples that combine the phase-space and geometric models to transform into observable space: surface-brightness profiles (Section 5), point measurements of the line-of-sight velocity (Section 6), and velocity profiles for spectra (Section 7). In Section 5 we compare the model qualitatively to simulations; in Section 6 we also demonstrate that g_s can be recovered within 50 percent or better for most cases. In Section 7 we discuss the advantages of integrated-light spectra over point sources and in Section 8 we summarize.

2 SIMULATIONS OF TIDAL CAUSTICS

To check that the analytical model represents the salient features of caustics, we will compare it with a set of simulations intended to gradually relax the assumptions of spherical symmetry and radial orbits that are fundamental to its derivation. The simulations are labeled A through D in order of increasing realism, for comparison with the analytical model; simulations B and C are actually two series that gradually increase the angular momentum in two different potentials. Table 1 summarizes the parameters of all the simulations.

In all four types of simulations we take a coordinate system in which z is the direction along the line of sight, centered on the host galaxy. In this case the x – y plane is effectively the plane of the sky, and will be used to represent the view of the tidal debris that we would observe. For this work we assume that the angular size of the galaxy is small enough that we can ignore the slight difference between the z coordinate and the radial line of sight (spherical projection effects). Then, for simulations A through C, we choose the initial position and velocity vectors for the satellite galaxy to lie mainly in the sky plane, so that the resulting shells are symmetric with respect to $z = 0$. This is the orientation, with respect to our line of sight, in which the shell edges appear sharpest and the kinematic signature is least complicated. In simulation D we relax this assumption.

Simulation A (Figure 1, top left) is the base case that fulfills all the assumptions in the derivation: a simulation of the disruption of a self-bound satellite in a static, spherical potential on a purely radial orbit ($L_{\text{COM}} = 0$). The series of simulations labeled B uses the same potential but gradually increases the orbital angular momentum, parameterized by the angle θ between the initial position and velocity vectors of the center of mass. $\theta = 0$ corresponds

to a radial orbit; $\theta = 90^\circ$ is a circular orbit. Using this parameter, the angular momentum is $L_{\text{COM}} = L_{\text{circ}} \sin \theta$, with $L_{\text{circ}} \equiv r_{\text{COM}} v_c(r_{\text{COM}})$. An example with a moderate amount of angular momentum, $\theta = 20^\circ$, is shown in the top right panel of Figure 1.

Series C uses a highly flattened, axisymmetric potential and sets the initial center-of-mass velocity vector of the satellite at 45 degrees to the plane of symmetry (the x - y plane) for maximum sensitivity to the nonsphericity of the potential. The amount of orbital angular momentum is adjusted as in series B, again parameterized by the angle θ . The result with $\theta = 0$ is shown in the lower left panel of Figure 1. This set of simulations is intended to exaggerate the effect of departures from spherical symmetry, thus stretching the capabilities of the model to the limit. Indeed, we find in this case that caustics selected in (r, v_r) , coordinates which no longer reflect the symmetry of the system, include both material passing through a turning point in R and material reaching a turning point in z . This further complicates the kinematic signatures for this case, but the projected distribution in the sky plane still features shells.

Simulation D (Figure 1, bottom right) differs from the other simulations in that it was not a product of systematic experiments, but is instead an N-body model of a real set of caustics (the Andromeda Giant Stream and associated shells) in a realistic, though still static, galactic potential (a mass model of M31). The best-fit model obtained by Fardal et al. (2007) includes two caustics produced by the nearly-radial encounter of an intermediate-mass satellite galaxy. This simulation is included in lieu of actual data so that we can compare the full phase-space information inferred by the N-body model with our analytic expressions, while emphasizing that these more realistic conditions can still give rise to suitably symmetric caustics.

3 A UNIVERSAL DENSITY PROFILE

In previous work (Sanderson & Bertschinger 2010) we derived a simple analytical form for a one-dimensional caustic formed from a system with initial random velocities drawn from a Maxwellian distribution (Figure 2, left panel). In the case where this system is a small, gravitationally self-bound satellite galaxy falling radially into the center of a static host galaxy, the one-dimensional density profile of the caustic, as a function of the galactocentric radius r , can be written in terms of four physical parameters δ_r , r_s , κ , and f_0 :

$$\rho(r) = \frac{f_0}{\sqrt{2\pi\kappa}} \sqrt{|r - r_s|} e^{-(r - r_s)^2 / 4\delta_r^2} \mathcal{B} \left[\frac{(r - r_s)^2}{4\delta_r^2} \right] \quad (1)$$

where \mathcal{B} is a piecewise combination of modified Bessel functions of the first kind:

$$\mathcal{B}(u) = \begin{cases} \frac{\pi}{2} \left[\mathcal{I}_{-1/4}(u) + \mathcal{I}_{1/4}(u) \right] & r \leq r_s \\ \frac{\pi}{2} \left[\mathcal{I}_{-1/4}(u) - \mathcal{I}_{1/4}(u) \right] & r > r_s \end{cases} \quad (2)$$

In Equation (1), δ_r is the characteristic width of the caustic surface, which depends on the initial velocity dispersion of the dwarf galaxy that created the caustic. A perfectly “cold” distribution, in which all particles had the same initial energy, would give rise to a true caustic with zero width and infinite density. We refer to caustics in a less rigorous sense: they have finite density and width but still exhibit a

large local density enhancement thanks to the same dynamical processes. The width δ_r is related, by conservation of phase space volume, to the phase space distribution of the satellite, but the relationship is not straightforward. The volume of phase space that is conserved is that of the material *after* it has been stripped from the satellite at pericenter. For such a radial orbit this volume can be roughly related to the pre-stripping phase space distribution of the satellite material via impulse approximation methods, but this involves further assumptions about the nature of the satellite. Likewise, the methods described in Helmi & White (1999) could in principle be used to relate the phase space volume to the width δ_r , but in practice many other factors (including projection effects, angular momentum, and departures from spherical symmetry in the potential) can increase δ_r , further complicating the relationship with the nature of the original satellite and increasing the number of assumptions necessary to obtain what amounts to a lower limit on the phase space density. For this work, we wish to remain as model-independent as possible with respect to the interacting galaxies, so we will treat δ_r as a free parameter in the remainder of this paper.

The radius of the caustic surface is denoted r_s . It is close to, but not equal to, the radius of peak density, r_{max} , because of the thickness of the caustic, induced by the nonzero velocity dispersion of the material. The peak location r_{max} and the peak density ρ_{max} may be determined by solving $d\rho/dr = 0$ numerically (necessary because of the Bessel functions). The peak radius r_{max} is related to the caustic radius r_s by

$$r_{\text{max}} = r_s - 0.765 \delta_r, \quad (3)$$

and the density at this location is

$$\rho_{\text{max}} = 1.021 f_0 \sqrt{\frac{\delta_r}{\kappa}}. \quad (4)$$

The curvature κ measures the shape of the stream near r_s in the (r, v_r) projection:

$$\kappa = -\frac{1}{2} \frac{d^2 r}{dv_r^2} \bigg|_{r_s}. \quad (5)$$

Because by definition the caustic surface is located where the distribution is vertical in the projection shown in the right panel of Figure 2, near the caustic its shape may always be approximated by a quadratic function:

$$r = r_s - \kappa(v_r - v_s)^2, \quad (6)$$

where v_s is the radial velocity at the caustic surface. Although all the material forming the caustic is near apocenter, the net radial velocity at the caustic surface is not necessarily zero because the spatial overlap of the orbits of individual stars in the stream depends on the distribution of their orbital periods, which in turn depends on the distribution of their energies.

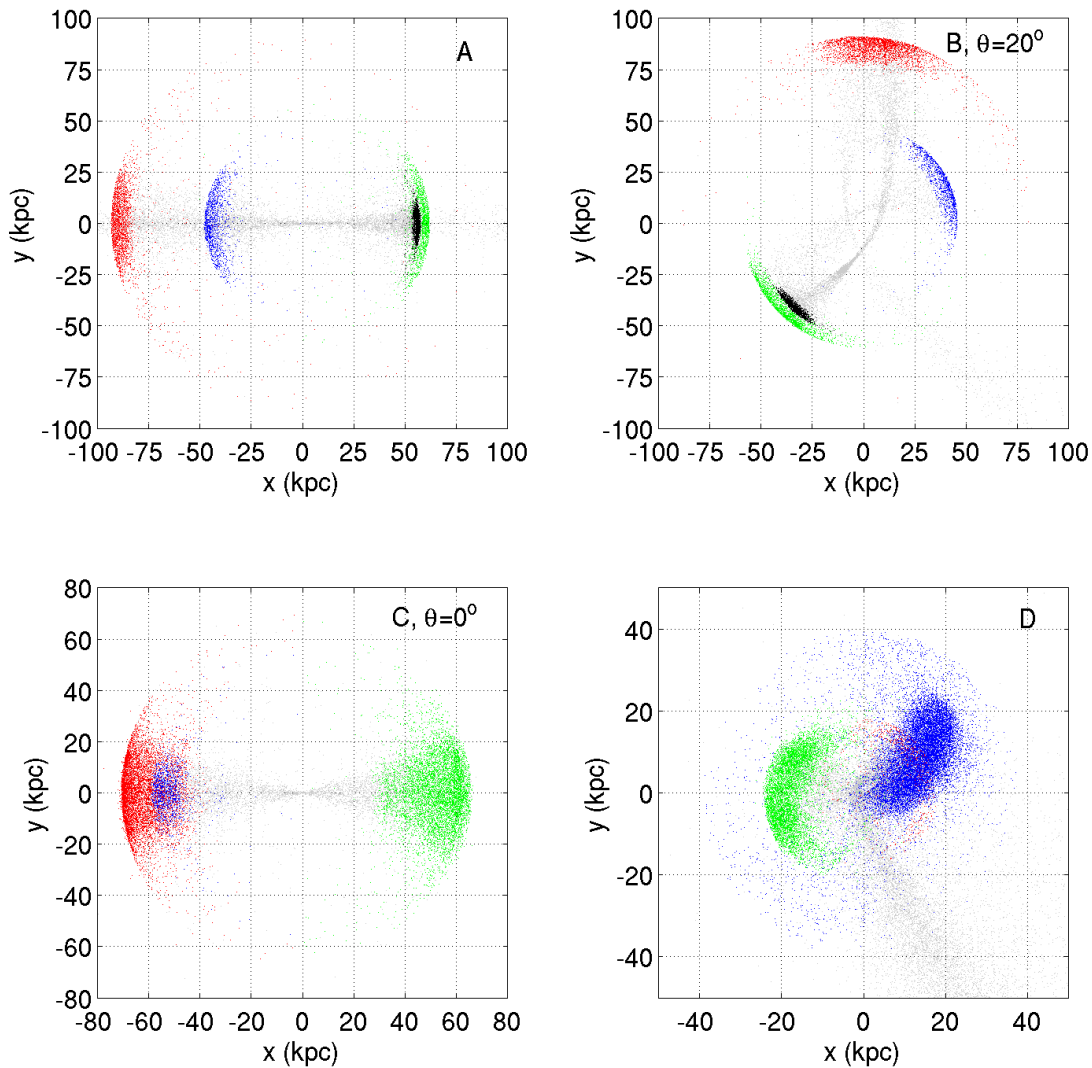
We can relate κ to the gravitational force at r_s by differentiating the energy equation evaluated at the caustic surface. For a purely radial orbit in a spherical galactic potential $\Phi(r)$, we have

$$\frac{d}{dr} \left(E = \frac{1}{2} v_r^2 + \Phi(r) \right) \rightarrow v_r \frac{dv_r}{dr} = -\frac{d\Phi}{dr} = g(r) \quad (7)$$

for the first derivative. If we invert the derivative in this

Table 1. Information about the four simulations whose results are presented in this work. See text for additional discussion. Expressions for the potential types in the table are given in Appendix A.

Name	Potential type	Potential parameters	Satellite mass ($\times 10^8 M_\odot$)	Initial \mathbf{r}_{COM} (kpc)	Initial \mathbf{v}_{COM} (km s $^{-1}$) $v_c \equiv v_c(r_{\text{COM}})$
A	Spherical Isochrone	$M = 2.7 \times 10^{12} M_\odot$, $b = 8.0$ kpc	2.2	(-40, 0, 0)	$(v_c, 0, 0)$
Series B	Spherical Isochrone	$M = 2.7 \times 10^{12} M_\odot$, $b = 8.0$ kpc	2.2	(-40, 0, 0)	$(v_c \cos \theta, v_c \sin \theta, 0)$, $\theta \in \{10^\circ \dots 90^\circ\}$
Series C	“Flattened” Isochrone	$M = 2.7 \times 10^{12} M_\odot$, $b = 8.0$ kpc, $q = 0.7$	2.2	(-40, 0, 40)	$(v_c \cos \theta, v_c \sin \theta, -v_c)/\sqrt{2}$, $\theta \in \{0^\circ \dots 90^\circ\}$
D	see Geehan et al. (2006)	$M_{\text{M31}}(< 125 \text{ kpc}) = 7.3 \times 10^{11} M_\odot$, $r_{\text{halo}} = 7.63$ kpc; see Table 2 of Fardal et al. (2007) for complete parameters	22	(-34.75, 19.37, -13.99)	(67.34, -26.12, 13.50)

**Figure 1.** Views of the sky (x - y) plane for the four types of simulations used in this work. In each simulation, caustics selected in $r-v_r$ space (shown in different colors) reflect the shell features seen in the plane of the sky. Simulations A-C have the same mass ratio of about 10^{-4} ; simulation D has a mass ratio of about 3×10^{-3} .

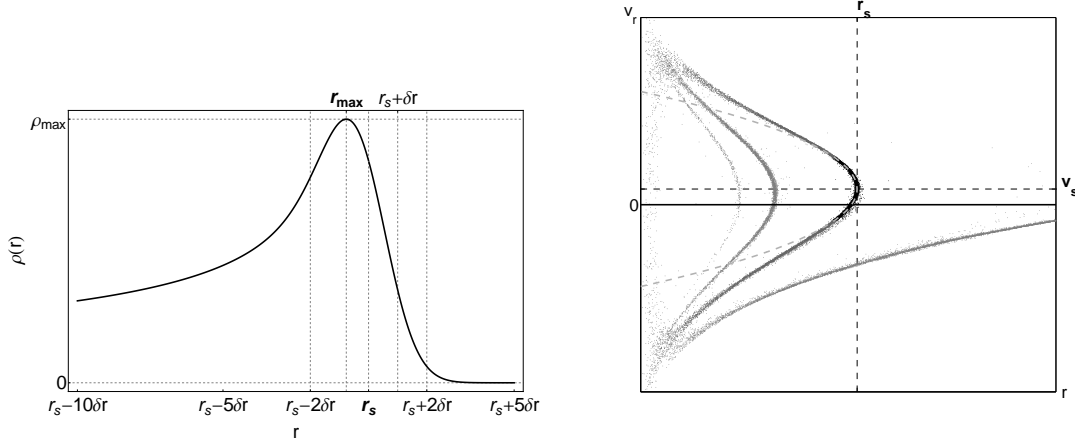


Figure 2. Left: Universal caustic form for radial infall from a population with initial Gaussian velocity dispersion. r_s and δ_r are defined and formulae for the peak radius and density are given in the text. Right: Phase space distribution taken from simulation D. Near the peak of one of the caustics (black), the phase space distribution follows the curve given by Equation (6) (dashed line), where v_s is the radial velocity of particles at the caustic surface. κ is defined in the text.

expression and differentiate with respect to v_r , then express the first derivative in terms of g , we find that

$$\kappa = -\frac{1}{2g(r_s)} \left[1 - \frac{v_s^2}{g(r_s)^2} \frac{dg}{dr} \Big|_{r_s} \right]. \quad (8)$$

We can suppress the dependence on the tide, dg/dr , by using Equation (8) to obtain a lower limit on κ instead of an equality, under certain assumptions about the mass distribution. It can be shown from the Poisson equation that dg/dr is negative for any spherical density distribution that satisfies the relation

$$\frac{M(< r)}{\frac{4}{3}\pi r^3} \equiv \bar{\rho}(r) < \frac{3}{2}\rho(r), \quad (9)$$

at the radius of interest. For a power-law density distribution, $\rho \propto r^\gamma$, this condition is satisfied for all $\gamma < -1$. Observational evidence and simulations of dark halos both indicate that the density almost certainly falls off faster than this at most radii, especially in the outer regions where shells are usually found. For example, the log-slope of the density of the Milky-Way-like halos of the Aquarius project satisfies this condition everywhere except in the innermost bin of the highest-resolution simulation (Springel et al. 2008). If we assume $dg/dr < 0$, then the second term in Equation (8) is always positive, so that the quantity in brackets is always larger than 1. The Poisson equation also requires that $g < 0$ at all r for a spherical potential, since there is no negative mass, implying that κ is positive. Thus neglecting the term involving the derivative of g turns Equation (8) into a lower limit on κ :

$$\kappa \geq \frac{1}{2|g(r_s)|} \quad (10)$$

For most reasonable models of the halo the neglected term is much less than 1, given that v_s is close to zero and the tidal forces are weak at large radius (where shells are most easily observed). Neglecting the gravity gradient term gives

$$\kappa \approx \frac{1}{2|g(r_s)|}, \quad (11)$$

relating the curvature of the phase-space stream directly to the gravitational force exerted by the host galaxy.

3.1 The effect of nonzero angular momentum and deviations from spherical symmetry

If the satellite galaxy had some initial orbital angular momentum L , the energy equation (7) is no longer correct, so measuring g by using the approximate relation (8) will have additional error. In this case, following the same derivation for κ while including nonzero L (but still assuming a spherical potential) leads to the relation

$$\kappa = -\frac{1}{2g_s} \left(1 + \frac{L^2}{r_s^3 g_s} \right)^{-3} \left[1 - \frac{v_s^2}{g_s^2} \frac{dg}{dr} \Big|_{r_s} + \frac{L^2}{r_s^3 g_s} \left(2 + \frac{3v_s^2}{r_s g_s} + \frac{L^2}{r_s^3 g_s} \right) \right], \quad (12)$$

where $g_s \equiv g(r_s)$. The overall $1/2g$ dependence is preserved, and the first two terms in the square brackets correspond to those in Equation (8). However, there are now new terms that all depend on the dimensionless quantity

$$-\frac{L^2}{r_s^3 g_s} \equiv \epsilon_L = \frac{L^2}{r_s^2 v_c^2}, \quad (13)$$

which compares the orbital angular momentum to that of a circular orbit at r_s , since $v_c(r_s) = \sqrt{r_s |g_s|}$ is the circular velocity at r_s . The minus sign is included in the definition to ensure a positive ϵ_L , since g_s is negative. When the orbit is purely radial, $\epsilon_L = 0$; when it is circular, $\epsilon_L = 1$.

Solving Equation (12) for g gives a complicated dependence on the angular momentum that can be simplified by expanding in terms of ϵ_L and its counterpart for the tidal force,

$$\epsilon_T \equiv -\frac{v_s^2}{g_s^2} \frac{dg}{dr} \Big|_{r_s} = -\left(\frac{v_s}{v_c} \right)^2 \left(\frac{r_s}{v_s} \right)^2 \frac{dg}{dr} \Big|_{r_s}, \quad (14)$$

and requiring that both be much less than 1. To first order

in both quantities,

$$g(r_s) \approx -\frac{1}{2\kappa} \left[1 + \epsilon_T - \epsilon_L \left(1 + \frac{6\kappa v_s^2}{r_s} \right) \right] + \mathcal{O}(\epsilon_L \epsilon_T). \quad (15)$$

The leading-order change in κ due to the angular momentum is at the L^2 level, so we expect that Equation (11) will remain a decent approximation even for satellites on fairly non-radial orbits. Furthermore, the corrections for tides and angular momentum have opposite signs, so at intermediate L we may expect help from competing error terms.

In practice we will not be able to make even first-order corrections for either tides or angular momentum, since one requires independent measurement of the gravity gradient and the other requires knowing the initial angular momentum of the satellite. However, with numerical experiments we can test how sensitive the phase-space shape is to non-radial orbits. To do this, we obtain κ from fits to the projected (r, v_r) distribution for shells in the simulations of Series B. We first estimate g using Equation (11), then progressively correct this estimate at first order according to Equation (15), taking first tides, then angular momentum into account. The simulations in the series form four caustics at different radii, and we obtain an independent estimate from each.

As seen in the left panel of Figure 3, the spherical approximation still recovers the gravitational force to within about 20 percent, even for relatively large amounts of angular momentum (up to half the maximum value). This is due to the quadratic dependence of ϵ_L on L and the partial cancellation of ϵ_T and ϵ_L . In some cases the combined first-order corrections for tides and angular momentum are sufficient to eliminate nearly all the error, while in others the correction is less good, especially at larger L . This is because the average L of material in each shell is not necessarily equal to L_{COM} , which was used to calculate the correction. Since ϵ_L is quadratic in L this difference is magnified at larger L .

We also see that the difference between a system that has less than 20 percent error using the radial approximation (Figure 3, center panel) and one that has much more (Figure 3, right panel) coincides with the difference between a shell-like and a stream-like morphology. The example with more angular momentum still has a sharp edge at the apocenter of each caustic, but the morphology of the debris is otherwise more stream-like than shell-like. Deviations from spherical symmetry in the potential will populate a larger proportion of orbital planes per unit L thanks to precession, and this should further accentuate the differences between low- L systems and those with higher L . Projection effects will also increase these differences; the two systems shown here are at an ideal viewing angle where the shells look the sharpest. We conclude that identifying shells (consisting of a fan shape and a sharp outer edge) by eye as low- L systems is probably a sufficiently strict selection criterion for applying the radial-orbit approximation to obtain estimates of g_s .

Deviations from spherical symmetry in the potential can also affect the ability to recover g_s . Out of the plane of symmetry, orbits in an axisymmetric potential will precess, thickening the phase space stream perpendicular to the (r, v_r) plane. Furthermore in an axisymmetric potential, the spherical radius r and its conjugate velocity v_r are not the coordinates in which the problem is separable, so there is no guarantee that the analysis developed here will still be ap-

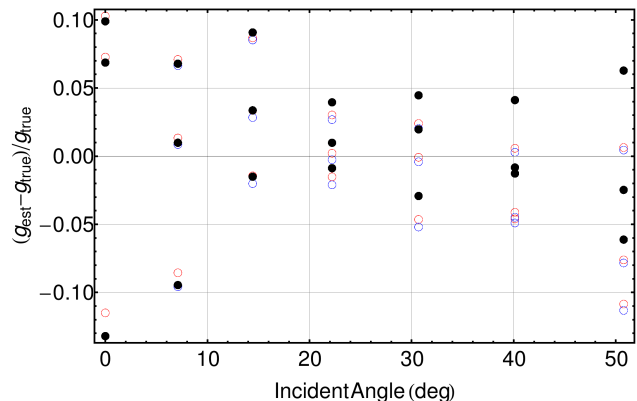


Figure 4. As in the left panel of Figure 3, but for simulations in series C (a highly flattened potential).

plicable to non-spherically-symmetric systems. To see how well we can do for a significantly flattened system, we repeat the process used to obtain Figure 3 on Series C. The results are shown in Figure 4. Equation (15) does not include the effect of precession, so the error prediction from the spherical analysis does not fully correct the estimate, but we can still recover g within 20 percent for the cases shown. We conclude that although we expect galactic potentials to be flattened, the inaccuracy from assuming spherical symmetry is not large enough to prevent us from measuring g .

4 A SIMPLE PHASE-SPACE DISTRIBUTION

Caustics have a simple structure in phase space as well as a simple density profile, thanks to the fact that the material in them was initially compact in phase space. The preservation of this initial small phase-space volume means that stripped material will spread out in a thin stream through phase space; thus the phase space distribution at late times is nearly one-dimensional. In the case of the caustics formed by a nearly radial encounter, symmetry tells us that the stream flows mainly along two of the six available coordinate directions: galactocentric radius r and radial velocity v_r . This is why, following Equation (6), we can construct a preliminary model of the caustic phase space distribution as a one-dimensional function of the six-dimensional phase space coordinates (\vec{x}, \vec{v}) by ignoring everything but these two coordinates:

$$f(\vec{x}, \vec{v}) \propto \delta[r_s - r - \kappa(v_r - v_s)^2]. \quad (16)$$

Normalizing this distribution so that the integral over a shell is unity results in the expression

$$f(r, v_r) = \frac{15\sqrt{\kappa}}{16r_s^{5/2}\Omega_s} \delta[r_s - r - \kappa(v_r - v_s)^2], \quad (17)$$

where Ω_s is the solid angle spanned by the shell. The derivation of this expression is given in Appendix B.

Unlike the model of the density profile, this model for the phase space distribution does not self-consistently take the energy spread into account, since it will also result in a spread of the phase space distribution around this line. If Equation (17) is integrated over v_r , one obtains the limit of

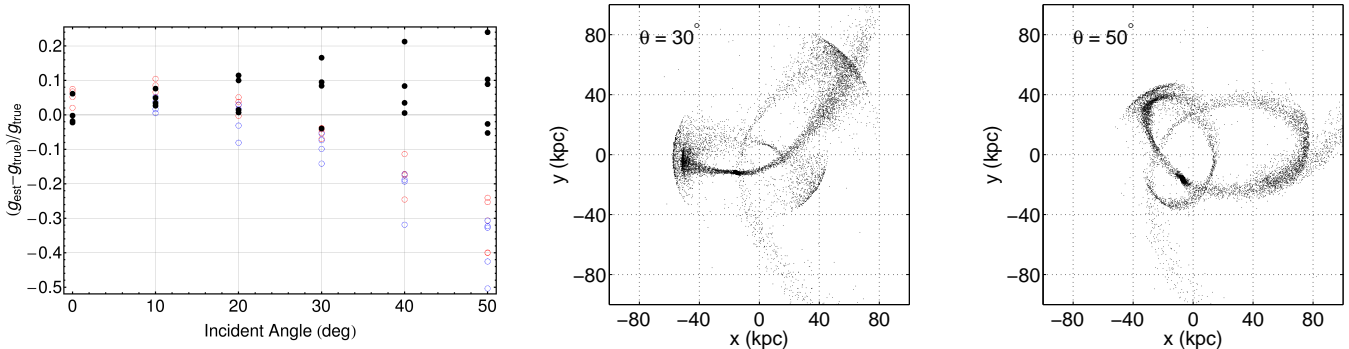


Figure 3. Left: the fractional error in recovering g remains small even at fairly large values of the orbital angular momentum (proportional to $\sin \theta$). The blue open circles show the relative error from using Equation (11) to estimate g . The red open circles include the first-order correction for tidal forces from Equation (15); the solid black points correct for both tides and angular momentum at first order using the same equation. Center and right: the tidal features at $\theta = 50^\circ$ (right), when the error on g_s exceeds about 20 percent, still exhibit a sharp outer edge but are morphologically distinct from fan-shaped shells with lower angular momentum (e.g. at $\theta = 30^\circ$, center).

Equation (1) as $\delta_r \rightarrow 0$; in other words, a piecewise function proportional to $1/\sqrt{r_s - r}$ for $r < r_s$ and zero beyond the caustic radius. However, we can obtain a profile consistent with Equation (1) in a manner analogous to the way in which Equation (1) was obtained, by assuming that the caustic is made up of many particles with slightly different caustic radii, normally distributed around the average r_s . This leads to a Gaussian form for the distribution function in r and v_r ,

$$f(r, v_r) = \frac{15}{16r_s^{5/2}\Omega_s} \sqrt{\frac{\kappa}{2\pi\delta_r^2}} \exp \left\{ -\frac{[r_s - r - \kappa(v_r - v_s)]^2}{2\delta_r^2} \right\} \quad (18)$$

A contour plot of this phase density in (r, v_r) space is shown in Figure 5. In Appendix C we give a derivation of this form, and show that integrating over all v_r retrieves the functional form of Equation (1).

As seen in Figure 5, this is not a perfect representation of the phase space distribution of a caustic: the density contours of the model do not exactly track the distribution in the caustic. In particular, we obtained Equation (18) by assuming that the mass in the stream is evenly distributed along it, when in reality there is no reason to assume this is the case. However, both the density profile (obtained by collapsing along the v_r axis) and the velocity profile (obtained by collapsing along the r axis) match simulated caustics quite well even when they are produced by mergers with significant angular momentum in highly nonspherical potentials. An example for v_r is shown in Figure 6. There is a slight difference between the completely flat distribution in v_r given by the model and the slightly peaked distribution seen in these two examples because the mass in the simulated stream is not necessarily evenly distributed, but the difference is remarkably small. So although the details of the phase space distribution are not perfectly encapsulated, this analytically tractable model can represent the main features of caustics formed from mergers quite far from the ideal case of a radial orbit in a spherical potential.

To keep the expression simple, we have also not taken into account the distribution of the material in v_θ and v_ϕ , instead assuming that all the material has zero angular momentum so $v_\theta = v_\phi = 0$. In reality, even a satellite whose center of mass is on a perfectly radial orbit will contain ma-

terial with a distribution of small angular momenta, and even in a spherical potential this will cover a distribution of orbital planes and tilt the distribution slightly out of the (r, v_r) plane. In this simplest case, the tilt results in a linear relationship between θ and v_θ and between ϕ and v_ϕ near the caustic. The relationship is more complicated when angular momentum is added or the potential is not spherical, thanks to the precession of the orbits in these cases. Unlike the caustic in the $r - v_r$ plane, the particular behavior in these subspaces is much more dependent on the details of the particular interaction. In the $r - v_r$ plane the main effect is to blur the caustic, leading to an over-estimate of the width δ_r compared to the true stream thickness. By using the projected $r - v_r$ space as our full phase space, therefore, we are mainly losing the ability to indirectly place a lower limit on the satellite's mass, as we discussed in Section 3. So this projection effect does not render too much information unusable.

5 IMAGES OF CAUSTICS

A few additional assumptions about the geometry of a shell will allow us to project the radial density profile given in Equation (1) onto the plane of the sky, allowing us in turn to obtain the parameters in that equation by fitting an image of a shell with the model. The projection process is greatly simplified by assuming that the material in the caustic is distributed evenly in angle, over some solid angle Ω_s . The edge of the caustic can then be approximated as a spherical segment that spans this solid angle, and the angular extent of the debris can be modeled most simply as a cone. Far from the caustic this is not necessarily a good approximation, but we are only concerned with the region within a few δ_r of r_s , where it appears to be adequate. In this model the shape and orientation of the debris in the caustic are therefore described by three parameters: the angles θ_s and ϕ_s of the cone relative to the line of sight z , and the opening angle α of the cone (Figure 7). The solid angle enclosed by the cone is $\Omega_s = 2\pi(1 - \cos \alpha)$. The angles are defined in the standard way for spherical coordinates, so that $(\theta_s, \phi_s) = (0, 0)$ corresponds to a cone opening directly away from the

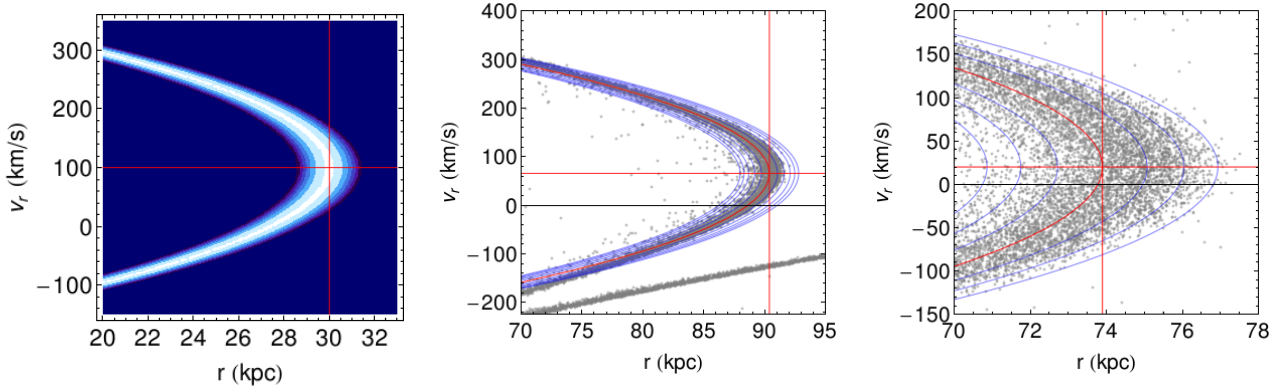


Figure 5. Contours of the model phase space density distribution for sample parameters (left); with fitted parameters overlaid on a caustic from simulation B; and with fitted parameters overlaid on a caustic from simulation C (right). In all panels the straight red lines cross at (r_s, v_s) . In the center and right panels the red parabola denotes the fit used to give κ , v_s , and r_s ; a fit to the density profile determined δ_r .

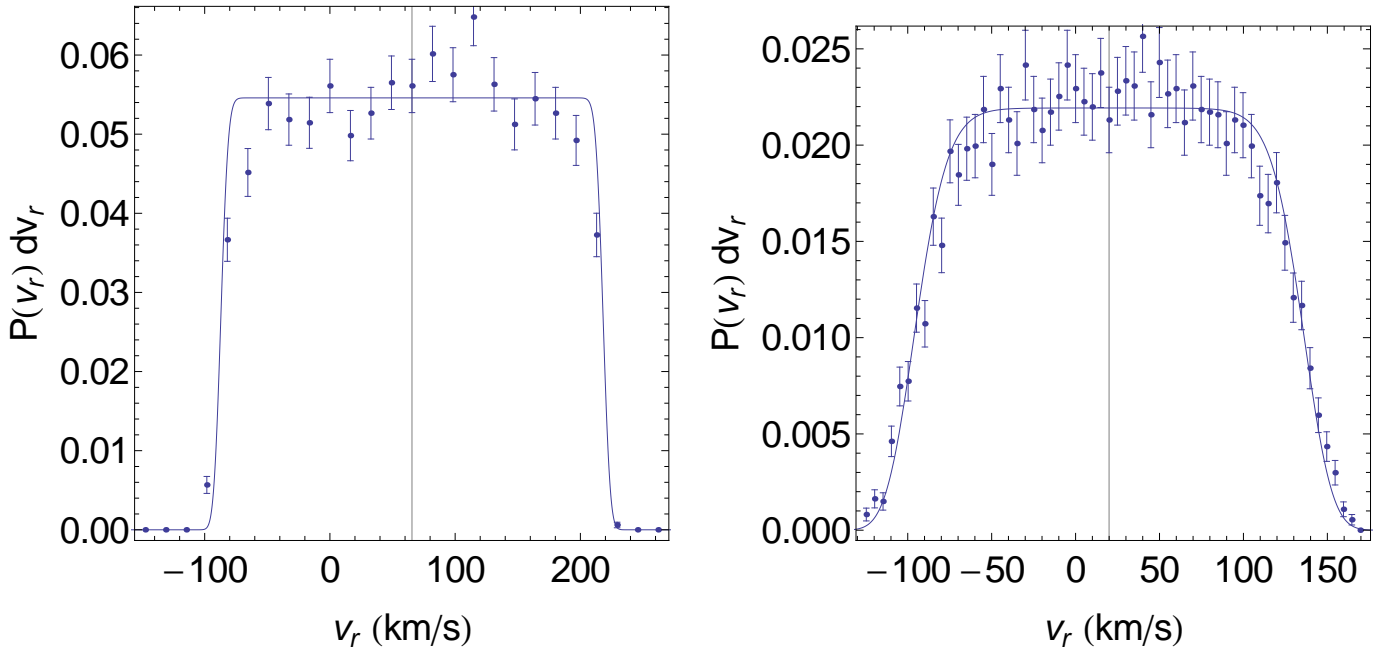


Figure 6. Probability distributions for the edges of two simulated caustics, compared to the phase space distribution integrated over $r > r_{\min}$. The normalization of the model is scaled by eye to match the data; the error bars indicate Poisson error on the number of counts per bin. Left panel: a caustic from simulation B with $r_s = 90.4$ kpc, $\delta_r = 0.6$ kpc and $r_{\min} = 81$ kpc. Right panel: a caustic from simulation C with $r_s = 73.9$ kpc, $\delta_r = 1.0$ kpc and $r_{\min} = 70$ kpc.

observer along the line of sight. If the caustic has a sharp edge in projection, this means that θ_s must be close to $\pi/2$, in which case α is close to the value measured in the plane of the sky.

For a given sky position (x, y) , we then obtain the surface density Σ by integrating the density distribution (1) along the line of sight z , for z values which are inside the

conical region occupied by the debris:

$$\begin{aligned} \Sigma(x, y) = & \mathcal{A} \int_{z_{\min}(\theta_s, \phi_s, \alpha; x, y)}^{z_{\max}(\theta_s, \phi_s, \alpha; x, y)} dz \sqrt{|\sqrt{x^2 + y^2 + z^2} - r_s|} \\ & \times e^{-\left(\sqrt{x^2 + y^2 + z^2} - r_s\right)^2 / 4\delta_r^2} \\ & \times \mathcal{B} \left[\frac{(\sqrt{x^2 + y^2 + z^2} - r_s)^2}{4\delta_r^2} \right], \end{aligned} \quad (19)$$

where we have defined $\mathcal{A} \equiv f_0 / \sqrt{2\pi\kappa}$. The limits of the integration depend on the geometric parameters of the shell and the sky position. We can obtain an analytic expression for these limits, for arbitrary parameters and position, by means of the standard cone equations. The full derivation,

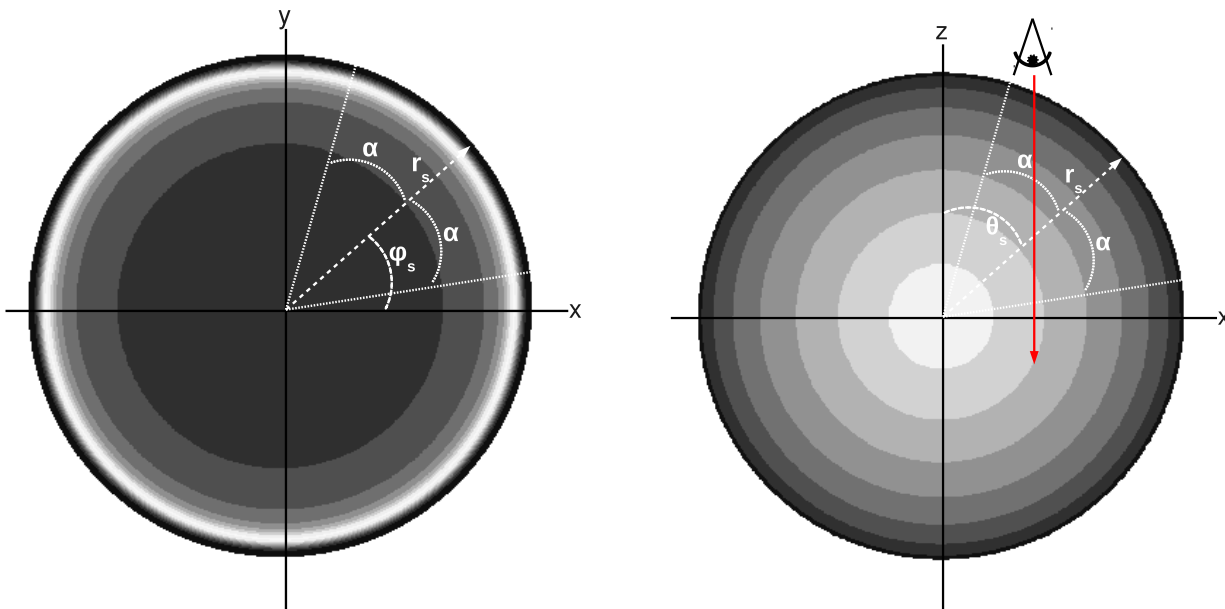


Figure 7. Left: the geometry of the shell and cone in the xy plane (the plane of the sky) in a slice through $z = 0$. The background shading is proportional to the density (lighter=denser), for a shell with radius r_s and thickness $r_s/50$. Superimposed are the angular limit of the shell α (dotted lines) and the orientation of the shell in the sky plane ϕ_s (dashed line). Right: the geometry of the shell in the xz plane (along the line of sight), sliced at $y = 0$. Here the background shading is proportional to the radial velocity, $|v_r - v_s|$ (lighter=higher value); at a given radius the radial velocity is double-valued at $\pm|v_r - v_s|$. The angular limit α (dotted lines) and the inclination θ_s with respect to the line of sight (dashed line) are superimposed. A line of sight (red solid line) with a given projected radius R , equal to x in this slice, probes a range of radial velocities and therefore line-of-sight velocities v_z . The range of v_z is limited by the values of v_s and κ according to Equation (23), but also by the angular extent of the shell and its orientation with respect to the observer.

and the method for calculating z_{\min} and z_{\max} , are given in Appendix D. The integral along the line of sight must be done numerically, but since the expressions for z_{\min} and z_{\max} are analytic in θ_s , ϕ_s , and α , standard minimization routines can be used to find best fit values by comparing the calculated and measured profiles. To this end we also provide derivatives of the integration limits with respect to the parameters in Appendix D.

Two examples of the result of this process are shown in Figure 8. We used the N-body realizations of two caustics from simulation D (left column) to construct two binned surface-density distributions (center column), which were then least-squares fit to Equation (19) to obtain the parameters $(r_s, \delta_r, \mathcal{A})$, which describe the radial density distribution, and the parameters $(\theta_s, \phi_s, \alpha)$, which describe the orientation and angular extent of the shell. The best-fit models are shown in the right-hand panel. Although the data clearly have smaller-scale complexity that is not accounted for in the model, the model is approximately consistent with the smooth features of the shell when comparing by eye. The surface-brightness fit is mainly used to provide an input geometry for fitting the kinematic data, from which g_s is recovered, so a model that roughly describes the extent, variation, and orientation of the shell is sufficient.

One significant exception is seen in the top row of Figure 8. Compared to the model, the caustic taken from the N-body simulation appears to have r_s smaller toward the top of the frame. Correlations between the energy and orbital phase of the material in the stream, combined with the presence of angular momentum, will introduce a variation in

the caustic radius with physical angle $r_s(\theta, \phi)$ which is not allowed for in the model. This variation can be comparable to δ_r . This discrepancy can be addressed by allowing r_s to vary, but since δ_r is mainly used as input into the kinematic models, this extra parameterization may not be needed.

The normalization \mathcal{A} involves the phase-space density f_0 and the gravitational force at the caustic g_s . If the model is fit to a surface-brightness map, the normalization obtained by the fit must be scaled by the estimated mass-to-light ratio of the debris to obtain the normalization of the mass-density profile. In principle, if we use Equation (11) as an approximation for κ and guess a reasonable value (or range) of f_0 , we can obtain an estimate of g_s :

$$g_s \approx -\frac{\pi \mathcal{A}^2}{f_0^2}. \quad (20)$$

However, due to the strong inverse scaling of g_s with f_0 , the estimate of the phase-space density has to be within better than a factor of about 3 just to get the right order of magnitude for the gravitational force. Furthermore f_0 refers to the fine-grained density, not the coarse-grained density probed by the stars, so it is unlikely that this approach will yield a reasonable constraint on g_s . We conclude that realistically, images alone can only constrain the combination $f_0 \sqrt{g_s}$. As we discuss in the next two sections kinematic data can separate this constraint into independent measurements.

Although it cannot produce an independent measurement of g_s , the surface-brightness fit plays a crucial role in the interpretation of kinematic data. Fitting the image models the geometry of the caustic, determining all the pa-

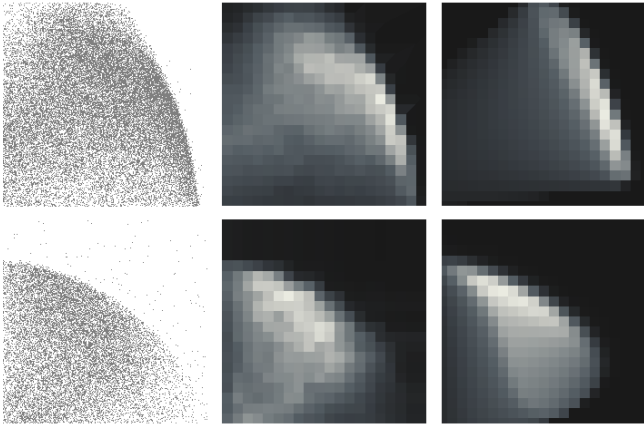


Figure 8. The binned “images” (relative brightness) generated for views of two caustics from simulation D (left: points from Nbody simulation; center: binned surface density) can be adequately modeled by a projection of Equation (1) onto the sky plane using a spherical cone to model the geometry as described in the text (right). The caustic parameters ($r_s, \delta_r, \mathcal{A}, \alpha, \theta_s, \phi_s$) used to generate the model image on the right in each case were obtained by a least-squares fit to the mock data in the center image.

rameters of the phase space distribution except for κ and v_s . These two parameters are necessarily convolved with the shell geometry in line-of-sight velocity measurements, so determining the geometry independently of the kinematics allows the most efficient use of the spectroscopic information which is so difficult to obtain. In the next two sections we discuss two ways that line-of-sight velocity data could be used to estimate g_s .

6 DISCRETE LINE-OF-SIGHT VELOCITY MEASUREMENTS OF CAUSTICS

For some shells, the line-of-sight velocities of point sources in the shell can be measured. These measurements probe the projection of the phase space distribution of Equation (18) onto the space of projected galactocentric distance R and line-of-sight velocity v_z . As was shown in Merrifield & Kuijken (1998), this projected distribution can have a distinctive, roughly triangular shape for shells that are oriented nearly edge-on with respect to our line of sight. The shape of the outer envelope of the profile, like the surface brightness profile, depends on a few parameters that can be fitted to obtain constraints on the gravitational field at r_s . More specifically, these parameters determine the relationship between the projected radius R and the maximum magnitude of the line-of-sight velocity at that radius, $v_{z,\max}$. Here we derive an expression for $v_{z,\max}(R)$, similar to the derivation in Merrifield & Kuijken (1998) but properly including the nonzero apparent outward velocity of the caustic. We then compare the result to simulations, and test how well the fit of this function can recover g_s .

6.1 Derivation of the envelope equation

A given projected radius R samples different three-dimensional radii r that correspond, via the phase space dis-

tribution, to different line-of-sight velocities v_z . For a satellite on a purely radial orbit,

$$v_z = v_r \frac{z}{r}, \quad (21)$$

so that for a given r and R , the line-of-sight velocity at the midline of the stream is described by

$$v_z = \left(v_s \pm \sqrt{\frac{r_s - r}{\kappa}} \right) \frac{\sqrt{r^2 - R^2}}{r}. \quad (22)$$

To find the maximum v_z at a given projected radius R we need to maximize this expression with respect to r , to find the physical radius r_e that contributes the highest line-of-sight velocities. Technically this results in a 6th-order polynomial expression for r_e ; however the approximation used by Merrifield & Kuijken, $r_e \approx (r_s + R)/2$, works quite well for the region near the shell.

Using the linear approximation for r_e and the first-order approximation for κ gives the formula for the maximum line-of-sight velocity as a function of projected radius alone, with $g(r_s)$ and v_s as parameters:

$$v_{z,\max} = \frac{\sqrt{r_s^2 + 2Rr_s - 3R^2} \left(\sqrt{g(r_s)(r_s - R)} + v_s \right)}{r_s + R} \quad (23)$$

Taking the limit $v_s \rightarrow 0$ and keeping only first-order terms in an expansion in R about r_s , we can recover the expression obtained by Merrifield & Kuijken:

$$v_{z,\max}^{\text{MK}} = \sqrt{\frac{g_s}{r_s}} (r_s - R). \quad (24)$$

We can derive an expression for the density in the projected phase plane, $f(R, v_z)$, by converting the full f to cylindrical coordinates and integrating over everything but R and v_z to project the distribution onto the plane of the sky and the line of sight. This means we must take the geometry of the shell into account, using the cone model derived in the previous section. Schematically, the full phase space distribution is limited to the cone by some function $\Theta(\theta_s, \phi_s, \alpha; \theta, \phi)$ in spherical coordinates. Since we are projecting onto the plane of the sky, we can separate the ϕ -dependence so that the cone limits are

$$\Theta(\theta_s, \phi_s, \alpha; R, z) \Theta(|\phi - \phi_s| \leq \alpha). \quad (25)$$

We have also assumed the velocity of the material is purely radial, and so v_θ and v_ϕ are both zero. In cylindrical coordinates this becomes

$$\delta(v_\phi = 0) \delta(v_R = v_z R/z). \quad (26)$$

Then the full phase-space distribution is

$$\begin{aligned} f(\vec{x}, \vec{p}) &= \mathcal{F}_0 \exp \left\{ - \frac{\left[r_s - \sqrt{R^2 + z^2} - \kappa \left(\frac{Rv_R}{r} + \frac{zv_z}{r} - v_s \right)^2 \right]}{2\delta_r^2} \right\} \\ &\times \Theta(\theta_s, \phi_s, \alpha; R, z) \Theta(|\phi - \phi_s| \leq \alpha) \\ &\times \delta(v_\phi = 0) \delta(v_R = v_z R/z) \end{aligned} \quad (27)$$

where

$$\mathcal{F}_0 \equiv \frac{15}{32\pi r_s^{5/2} (1 - \cos \alpha)} \sqrt{\frac{\kappa}{2\pi \delta_r^2}} \quad (28)$$

Now we must integrate over v_ϕ, v_R, ϕ , and z . All of these

Parameter	Value, Caustic 1	Value, Caustic 2
r_s (kpc)	92.8	47.7
δ_r (kpc)	0.32	0.15
v_s (km s ⁻¹)	67.	40.
$ g_s $ (km s ⁻¹ Myr ⁻¹)	1.2	4.0

Table 2. Parameters for the phase-space models of the two caustics shown in Figure 9, derived by fitting the full phase-space information as described in the text.

but the z integral are delta functions or step functions, and can be trivially evaluated:

$$f(R, v_z) = 2\mathcal{F}_0\alpha \int_{-\infty}^{\infty} dz \Theta(\theta_s, \phi_s, \alpha; R, z) \times \exp \left\{ - \frac{\left[r_s - \sqrt{R^2 + z^2} - \kappa \left(\frac{R^2 v_z}{r z} + \frac{z v_z}{r} - v_s \right) \right]^2}{2\delta_r^2} \right\} \quad (29)$$

The remaining step function limits the line-of-sight integral to points inside the spherical cone used to describe the shell's geometry; thus we can rewrite the integral by replacing the limits of integration with the functions of R derived in Appendix D (suppressing the parameter dependence for simplicity), just as we did to calculate surface brightnesses:

$$f(R, v_z) = 2\mathcal{F}_0\alpha \int_{z_{\min}(R)}^{z_{\max}(R)} dz e^{-\left[r_s - \sqrt{R^2 + z^2} - \kappa \left(\frac{R^2 v_z}{r z} + \frac{z v_z}{r} - v_s \right) \right]^2 / 2\delta_r^2} \quad (30)$$

The line-of-sight integral is then performed numerically to obtain the distribution for a given R and v_z .

6.2 Comparisons with simulations

To see how well the envelope equation and the projected distribution in R and v_z represent the main features of shells, we compare them to simulated tidal caustics. We first consider simulation A, which satisfies all the assumptions used to derive the models. The top panel of Figure 9 shows the system in the (r, v_r) projection of phase space, with two caustics identified in red and blue. The simulation provides us access to the full phase-space information of the shell: as a first test of the model's ability to represent the observed quantities, we can get the parameters of $v_{z,\max}(R)$ from the complete phase space information and compare them in the observed space (R, v_z) . To determine the parameters we fit Equation (6) to each caustic (solid cyan and magenta lines) to obtain values for r_s , v_s , and g_s ; we also obtain δ_r by fitting the radial density profile of each caustic (a histogram of the r coordinates of the colored particles) with Equation (1). We need δ_r because the spread around r_s is not taken into account by Equation (23), so the envelope should be shifted radially outward to the very edge of the distribution at $r_s + 2\delta_r$. The parameters are given in Table 2.

The center panel of Figure 9 shows the same system in the observed space (R, v_z) . For both caustics it is clear that Equation (23) (cyan/magenta) is a better description of the outer envelope than Equation (24) (green). However, some material still lies above the envelope described by the equation for v_{\max} , especially for the outer caustic, even though we have adjusted for the spread of the material around r_s .

This is a result of correlations in the perpendicular $\theta - v_\theta$ space that are specific to spherical potentials. The velocity v_θ varies linearly with θ along the stream thanks to the symmetry in this coordinate, is centered on zero near the caustic, and has a range of values comparable to v_s in the range of interest in R . For this highly symmetric system, this correlation causes a rounding of the outer envelope and leads to a slight overestimate of g . If we examine a case where the angular momentum is nonzero (Simulation B, Figure 10), we see that the addition of angular momentum breaks the symmetry in the $\theta - v_\theta$ plane, leading, counterintuitively, to an improvement in the fit.

The lowest row of Figure 9 shows the modeled projected density in the $R - v_z$ plane, Equation (30), for each of the two modeled caustics. The model captures the interior structure in the density distribution that is seen in the N-body distribution (second row), including significant density enhancement in the interior of the outer (red) caustic in the projected space. In practice, the number of measurements in this space is likely to be fairly small, probably insufficient to distinguish the structure of the distribution by eye. Because the outer envelope is not filled with a constant density of points, a sparse sampling of measurements in this plane will not give a good estimate of $v_{z,\max}(R)$. Fitting Equation (23) directly to an ensemble of measured line-of-sight velocities is therefore not a good way to determine the gravitational force. Instead, the measurements should be assigned probabilities for a given set of model parameters using Equation (30). Then the parameters, including g , can be recovered by maximizing the total likelihood.

Next we consider an example in which the potential has a realistic axisymmetric contribution from a disk: simulation D (Figure 11). (We will return to simulation C later in this section.) The caustics in simulation D combine three important differences from the highly symmetric situation in simulation A: they are not aligned exactly with the line of sight, they were formed with some small angular momentum, and the potential is not spherical. In the plane of the sky (leftmost panel), the outermost caustic, shown in blue, appears less sharp-edged than the next caustic, shown in green. As before, we fit the $r - v_r$ profiles of the caustics (middle panel) to obtain r_s and v_s ; in this case we also use the fitted value for κ which is not far from the true value of $g(r_s)$. In the projected space (right panel) we see again that the velocity profiles are not all symmetric around $v_z = 0$, but the displacement can be adequately modeled by a constant, and the distributions are fairly symmetric around an offset velocity axis. We will show momentarily that these are primarily projection effects. Second, the outermost caustic is rotated enough to affect the velocity profile, since the bulk of the material does not satisfy the approximation $R \approx r$ (or $\theta_s \approx \pi/2$) used in the derivation. Nonetheless, the envelope fits the material that does satisfy this approximation, and when the envelope is translated in R to encompass the bulk of the observed material (dashed black line) it is still a roughly accurate description. The difference between the very outer edge of the shell and the outer edge of the main part of the material is about 5 kpc, so the uncertainty in R will not result in a large error in g_s . This suggests that Equation (23), with some modifications, should be able to recover g_s for real shells.

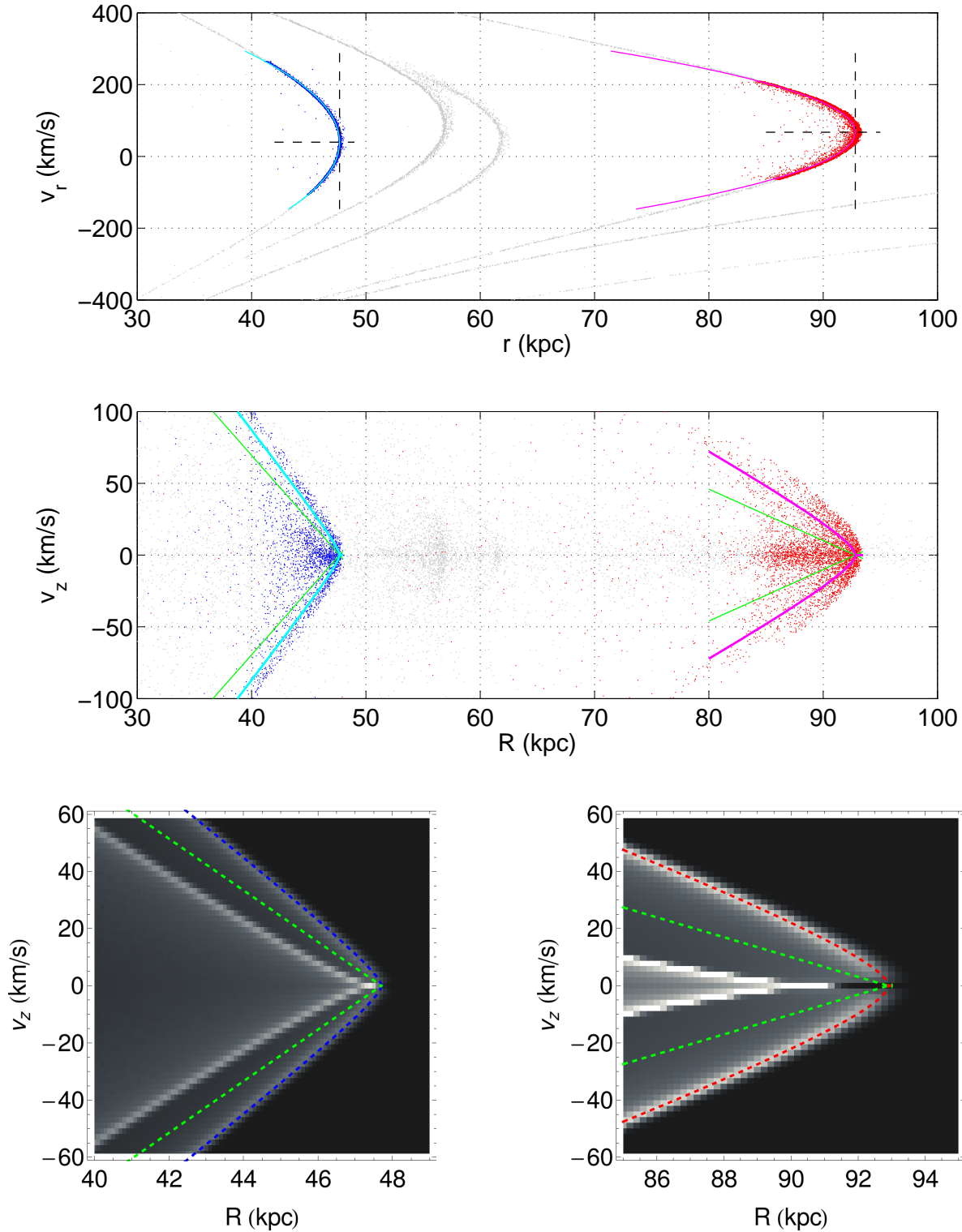


Figure 9. Line-of-sight velocity profiles and phase space distributions for two caustics from simulation A. The $r - v_r$ projection and the selected caustics are shown in the top panel (blue and red points) with quadratic fits overplotted (cyan and pink lines). The middle panel shows the projected line-of-sight velocities in the plane of the sky with Equations (24) (green) and (23) (cyan and pink) overplotted, using the values for r_s and v_s obtained from the fits in the top panel and $\kappa = 1/|2g(r_s)|$ for the input potential. The bottom panel shows the same two equations (green and blue/red dashed, respectively) superposed on the projected density, Equation (30), for the same two caustics.

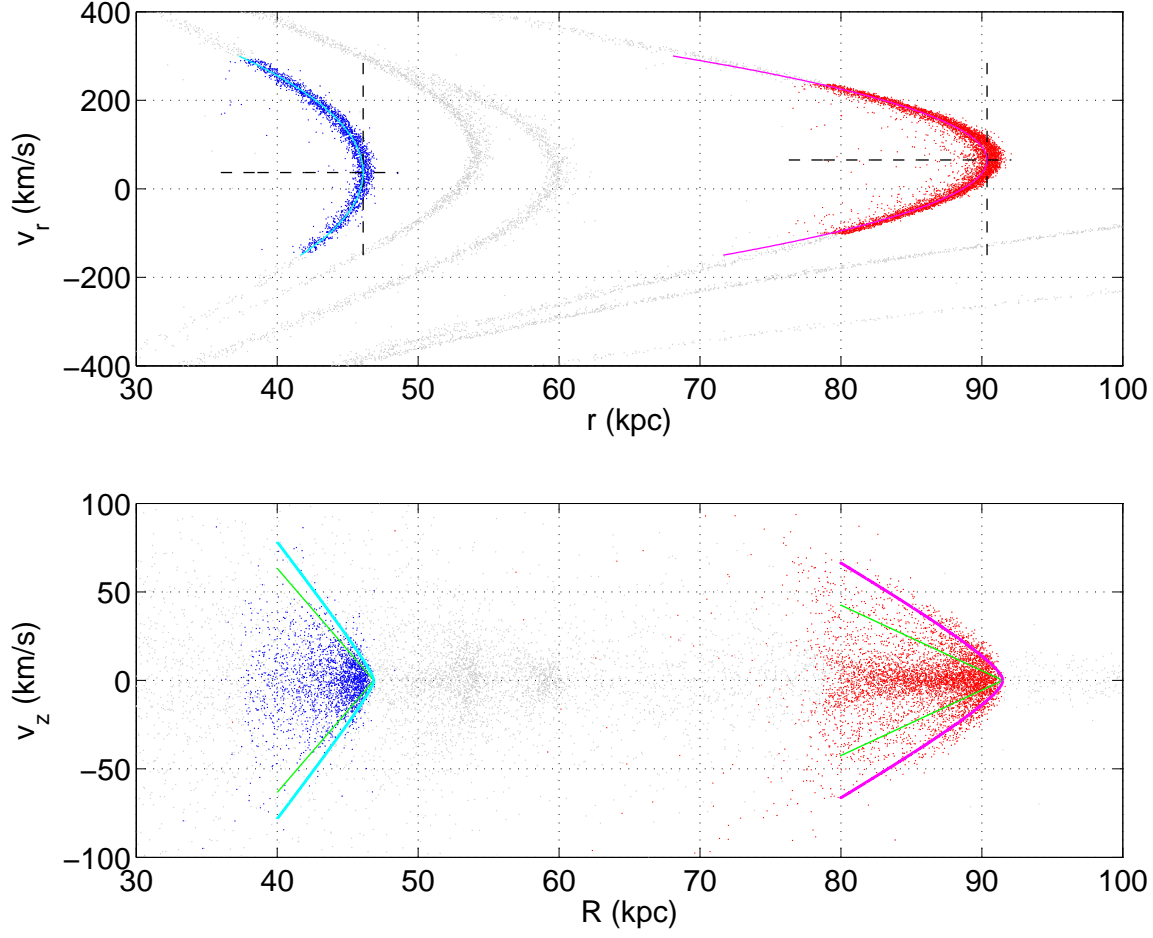


Figure 10. As in Figure 9, but using simulation B. The nonzero angular momentum actually improves the agreement of the model and simulation in the (R, v_z) plane somewhat.

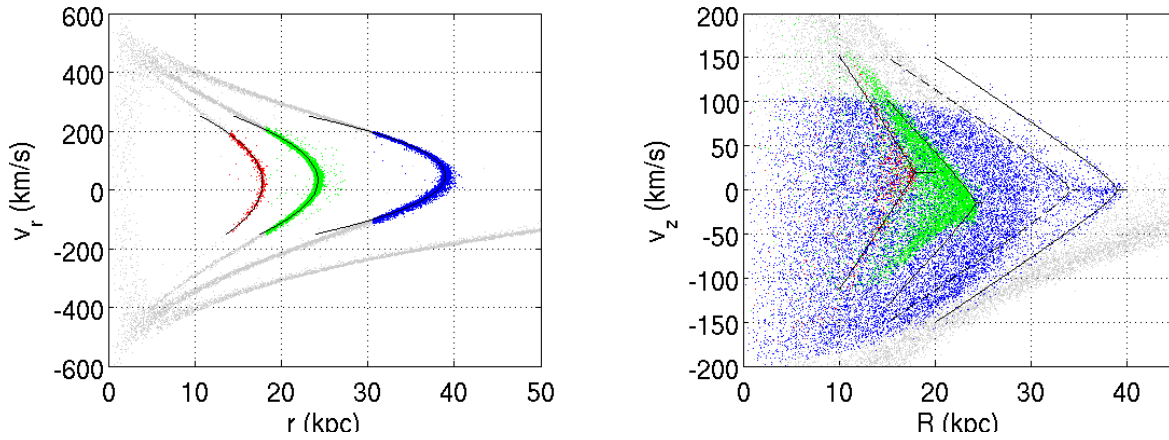


Figure 11. Views similar to Figure 9, but for simulation D. The red, green, and blue regions are the same as in Figure 1. The left panel shows the view in the $r - v_r$ plane with fits to the colored regions as black lines. The right panel shows the $R - v_z$ plane with envelopes calculated using Equation (23) and the values from the fits in the center panel; the two inner envelopes are translated by eye in the v_z direction to fit the symmetry axis. The dashed line is a translation in R of the envelope for the outermost caustic. See the text for further discussion.

6.3 Projection effects

The orientation of the shell with respect to the line of sight can affect the view in the (R, v_z) plane. In Figure 12 we show the effect of rotating the outermost caustics from simulations A and B from a perfectly symmetric orientation ($\theta_s = 90^\circ$) away from the observer ($\theta_s < 90^\circ$). All these examples have θ_s very different from 90° ; at inclinations closer to 90° there is not very much change to the distribution, but if the shells are nearly spherical (as these are) then even at relatively large inclinations they may still have a sharp edge in projection. We see that there are three main effects: the peak density is no longer on the v_z axis, the distribution is no longer symmetric about the v_z value defined by this point, and the internal structure seen in the distribution is smoothed out.

The distribution is offset in v_z because the average dot product with the shell expansion velocity v_s , and any net rotation in the line-of-sight direction, is now nonzero. In the figure we have modeled this shift by

$$v_{\text{offset}} = -\frac{1}{2} \left(v_s + \frac{L_{\text{los}}}{r_s} \right) \cos \theta_s, \quad (31)$$

where the first term accounts for the shift from the shell velocity and the second for the rotation. As shown in the figure by the dashed lines, this model is a good predictor of the vertical shift for spherically symmetric potentials. In the case of simulation B, the angular momentum is in the $-z$ direction so that the two terms nearly cancel. This also causes the lower half of the distribution to be selected instead of the upper half for this case, even though in both simulations the viewing angles are the same.

The asymmetry about the $v_z - v_{\text{offset}}$ axis is a combined effect of the inclination angle and the finite angular extent of the shell—essentially, the line of sight now probes an asymmetric distribution of velocities. However, it is clear from the figure that the “filled” side of the distribution still has the same envelope as the symmetric distribution. Rotating the caustic in the other direction (toward the observer instead of away from them, $\theta_s > 90^\circ$) will “fill” the other side of the distribution instead, but the maximum value of $|v_z - v_{\text{offset}}|$ still follows the model. Additionally, for caustics inclined to the line of sight the projection smoothes out the interior structure of the distribution somewhat, which could be an advantage when sampling this plane with discrete sources since the sampling is then more uniform in the populated part. In any case, the only asymmetry that the projection effect can produce is to cut out part of the filled area under the envelope. This may complicate efforts to fit the envelope, so an understanding of the geometry, especially the inclination angle θ_s , is important since kinematic measurements are unlikely to populate the whole (R, v_z) plane the way these simulations do.

6.4 Indications of a flattened potential

A significantly flattened potential like that of simulation C can complicate the shape of the (R, v_z) plane in ways that cannot be obtained by inclining a shell to the line of sight as discussed in the previous section. For extremely flattened potentials the kinematic plane can look quite different from the examples presented above, even if the caustics appear

shell-like in position space as they do in this case (see Figure 1). This example is intended to be especially tricky because the potential is not only extremely flattened, but face-on (the symmetry plane is the x - y plane) so that the degree of flattening could not be estimated from the galactic shape for this case.

The caustics from simulation C, shown in Figure 13, do form the characteristic structure (though significantly broader) in the (r, v_r) plane (top left), but the (R, v_z) projection (top right) now looks remarkably different. The caustics are not symmetric about $v_z = 0$, as expected for shells inclined relative to the line of sight, but additionally the slope of $v_{z, \text{max}}(R)$ no longer agrees with the model prediction based on a fit to (r, v_r) . The solid lines in the top right panel of Figure 13 show Equation (23) shifted by eye in v_z to the rough axis of symmetry of each shell, and show that this approximation is not a good description of the distribution in the projected phase space. This failure is due to a combination of effects. First, examining the phase space in projections consistent with the potential’s symmetries (Figure 13, bottom row) shows that especially for the outer (red) caustic, the material is actually passing through two caustics, one in R and one one in z . So whereas in a spherical potential v_z is single-valued with respect to z , thanks to this double caustic structure it is now double-valued. Furthermore, the caustic in (R, v_R) is by definition not probed by the line-of-sight velocity, where the spherical caustics had a component of v_r along the line of sight. In addition, we see from the range of z values in the lower right panel that the assumption that $R \approx r$ is no longer valid. As a result the dot product of the velocity with the line-of-sight direction is now primarily giving us information about the behavior of v_z , not the apparent caustic in (r, v_r) .

We include this example to demonstrate how measurements of the kinematics in caustics can diagnose a significant amount of flattening in the potential. The degree of displacement of the (R, v_z) distribution from the v_z axis shown in Figure 13 is quite large compared to those obtained by rotation in Figure 12; interpreted in the context of this model, one would have ventured that the distribution seems to indicate a large amount of angular momentum combined with a θ_s very different from $\pi/2$. However, the shells still look sharp in projection on the sky, which is inconsistent with a large inclination and unlikely for such a large L . The distribution in (R, v_z) very clearly does not follow the model envelope on either the positive or negative side, a type of asymmetry which cannot be replicated by adjusting the viewing angle, and includes extra lumps and curvature that the model does not. These signatures, whether probed by discrete measurements or integrated spectra, could thus potentially be used to diagnose departures of the potential from spherical symmetry.

6.5 Recovery of the gravitational force

We can also use the simulations presented here to quantitatively test the ability of this method to recover the value of g_s in the case of projected observables. For each caustic we need to fit the envelope of the points from the N-body simulation in the (R, v_z) plane. To do so, we bin the points in R and choose $v_{z, \text{max}}$ in each bin at the n th percentile in v_z , where n is usually between 95 and 100. We choose n as

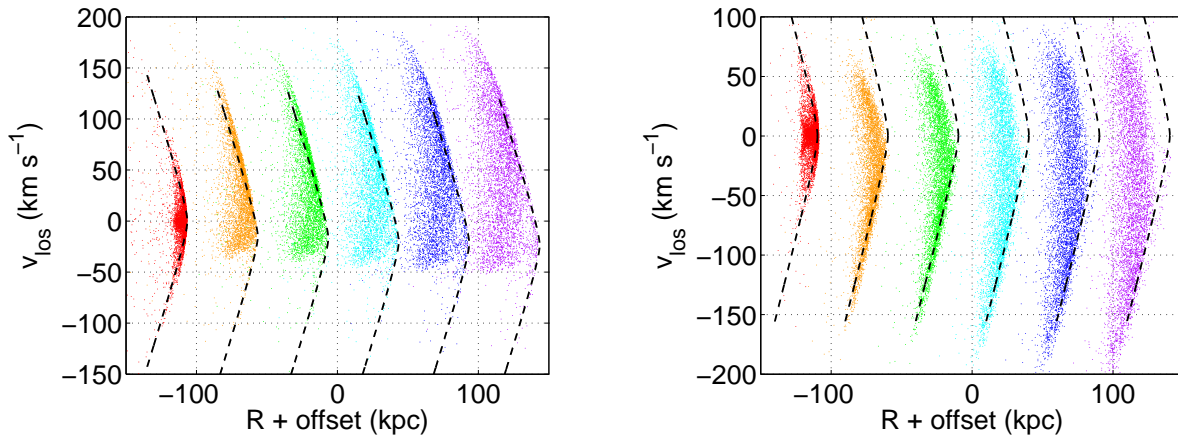


Figure 12. Effect of shell orientation relative to the line of sight on the (R, v_z) plane, in simulations A (left) and B (right). The colors red through purple indicate $\theta_s = \{90, 63, 60, 53, 49, 45\}^\circ$. The dashed black lines are the envelope derived in the text, translated in v_z by the amount defined in Equation (31). The v_z shift and asymmetry are results of taking $\theta_s \neq 90^\circ$, but the envelope still fits the more fully populated side of the distribution.

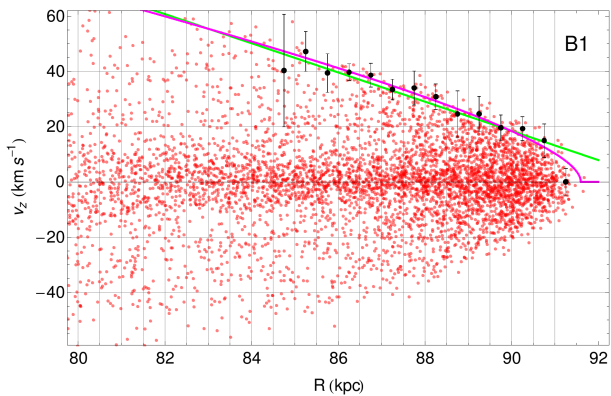


Figure 14. Example of envelope data and fits for the outer caustic from simulation B. The black points with error bars are obtained from the red points (the simulation particles) as described in the text; the green line is the fit to Equation (24) and the pink line is the fit to Equation (23). This is one of the less well-fit cases, as indicated by the overshoot on the front edge of the distribution, which leads to a larger uncertainty on the recovered g_s .

large as possible in each case so that we get the edge of the envelope but are not contaminated by single outlier points above it. We estimate the errors on these $v_{z,\max}$ points by looking at the difference in v_z between roughly the n and $n - 2$ percentile, to get a sense of the “fuzziness” of the edge in each bin. Finally, we add a point at the front edge of the distribution where $v_{z,\max}$ goes to zero. This method is not meant to be representative of how real data would be analyzed (for one thing, it is not likely that we would have thousands of real velocity measurements to work with) but was developed as a rough way to represent the envelopes of the particle representations of the simulated caustics. An example is shown in Figure 14. We then carry out weighted least-squares fits of Equations (23) and (24) to recover g_s , to test whether one form is better than another at measuring g_s and how close we can get to the input value.

The results, shown in Figure 15, show that fitting Equa-

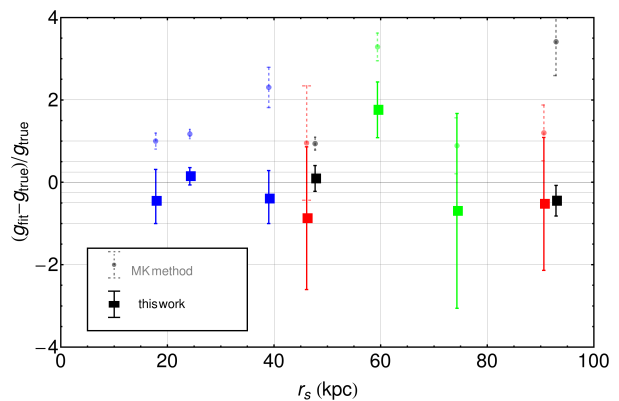


Figure 15. Recovered values of g_s for the caustics from simulations A-D pictured in Figures 9-11 (black=A, red=B, green=C, blue=D). Each caustic was fit twice, once using Equation (24) (points with dashed error bars) and once using Equation (23) (filled squares with solid error bars). The error bars are the projected 95 percent confidence intervals from the least-squares fits.

tion (24) (open squares) consistently overestimates g_s by large factors, up to a factor of 4 in some cases. Fits to Equation (23) (filled circles) are consistently better at recovering g_s , although in a few cases the enormous error bars indicate that this can be a more difficult fit to converge. The only cases in which both equations are more or less equally bad are the two caustics from simulation C (green points), which we do not expect to give a good result for the reasons discussed above. In most cases we can recover g_s within 50 percent using Equation (23).

7 SPECTROSCOPIC LINE-OF-SIGHT VELOCITY MEASUREMENTS OF CAUSTICS

Spectroscopic measurements of integrated light use a slit or a set of fibers to record the line-of-sight velocity profile at a given location in the plane of the sky. We can calcu-

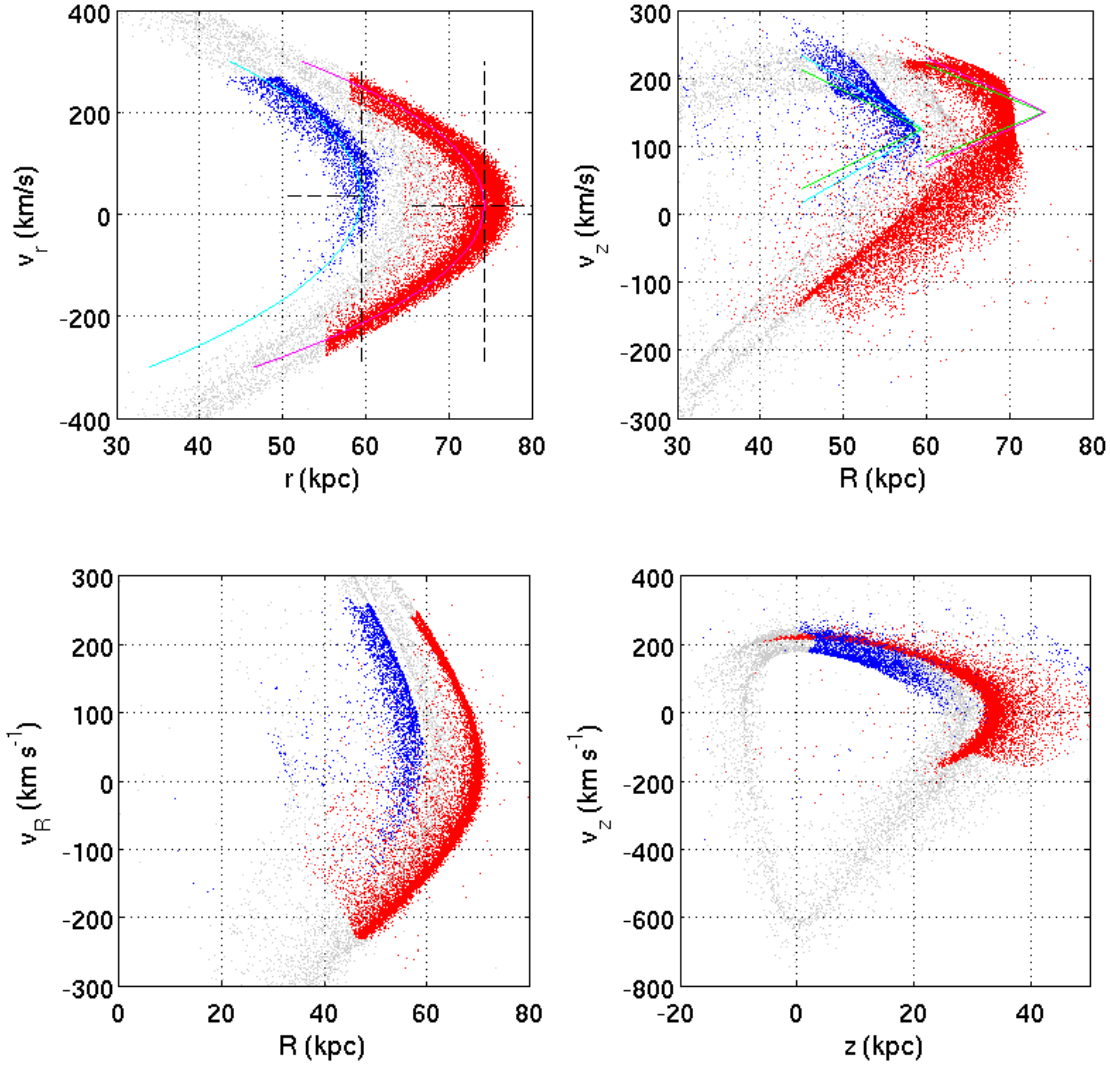


Figure 13. Top row: as in Figures 9 and 10, but for simulation C. Here the model, which assumes that the shell lies around the plane of the sky so that z is small, does not work even when translated in v_z to the axis of symmetry for each caustic. Bottom row: slices of the phase space in natural axisymmetric coordinates show that material is actually passing through two fold caustics at once, in R (left) and z (right). Also note from the z axis that the assumption that $r \approx R$ used to derive Equation (23) is not valid for this caustic.

late the expected velocity distribution in a slit or fiber from the phase-space distribution. As an example, we consider a single round fiber of the type that might be used in an integral-field unit. The center of the fiber is located at some sky location $(x_{\text{fib}}, y_{\text{fib}})$ and has a radius ρ_{fib} . We take these quantities to be in physical units (e.g. kpc) so that $g(r_s)$, embedded in the phase space distribution, still has reasonable units. To get angular units for the fiber values one has only to divide by D , the distance from the observer to the target. The supplied profiles can then be convolved using spectral simulator tools to produce the expected lineshape for fitting purposes.

For a circular fiber of arbitrary size, the velocity profile requires three integrals: over the line of sight, and over aux-

iliary variables ξ and η that span the area subtended by the fiber:

$$f(v_z) = 2\mathcal{F}_0\alpha \int_{-\rho_{\text{fib}}}^{\rho_{\text{fib}}} d\xi \int_{-\sqrt{\rho_{\text{fib}}^2 - \xi^2}}^{\sqrt{\rho_{\text{fib}}^2 - \xi^2}} d\eta \int_{z_{\text{min}}(\xi, \eta)}^{z_{\text{max}}(\xi, \eta)} dz \exp \left\{ - \left[r_s - \sqrt{R^2 + z^2} - \kappa \left(\frac{R^2 v_z}{r z} + \frac{z v_z}{r} - v_s \right)^2 \right] / 2\delta_r^2 \right\}, \quad (32)$$

where

$$R \equiv \sqrt{(x_{\text{fib}} + \xi)^2 + (y_{\text{fib}} + \eta)^2} \quad (33)$$

and

$$r \equiv \sqrt{(x_{\text{fib}} + \xi)^2 + (y_{\text{fib}} + \eta)^2 + z^2}. \quad (34)$$

For a different spectroscopic geometry, e.g. a slit instead of a fiber, one would simply alter the integration ranges of ξ and η in this expression.

To simplify the calculation we can assume that the angular size of the fiber is small compared to the angular size of the shell, so that it probes a range of physical R comparable to or smaller than the shell thickness δ_r . For many shells this is likely to be a sufficient approximation; for example, a 4" fiber has a fiber radius of about 0.5 kpc at a distance of around 25 Mpc, comparable to δ_r for the shells in our simulations (see Table 2). Assuming that each fiber probes a single line of sight allows us to replace the limits of the line-of-sight integral with their central values $z_{\min}^{\max}(x_{\text{fib}}, y_{\text{fib}})$, and the ξ and η integrals with $\pi\rho_{\text{fib}}^2$ times the central value of the integrand. This reduces Equation (32) to a single integral over the line of sight:

$$f(v_z) = 2\pi\rho_{\text{fib}}^2\mathcal{F}_0\alpha \int_{z_{\min}}^{z_{\max}} dz \exp\left\{-\left[r_s - \sqrt{R_{\text{fib}}^2 + z^2} - \kappa\left(\frac{R_{\text{fib}}^2 v_z}{z\sqrt{R_{\text{fib}}^2 + z^2}} + \frac{zv_z}{\sqrt{R_{\text{fib}}^2 + z^2}} - v_s\right)^2\right]^2 / 2\delta_r^2\right\}, \quad (35)$$

where

$$R_{\text{fib}} \equiv \sqrt{x_{\text{fib}}^2 + y_{\text{fib}}^2}. \quad (36)$$

For some lines of sight this distribution has four symmetric peaks, while others closer to the shell have a double peak (Figure 16). The shape of the profile depends on the distance from the shell edge, the shell thickness, the orientation of the shell relative to the line of sight, and the angular span of the shell. The distance from the shell edge determines whether two peaks or four are observed: for the line of sight in Figure 7, ignoring the angular cutoffs momentarily, four peaks will be observed because the line of sight probes radii far from the shell edge, where the two values of $\pm|v_r - v_s|$ are widely separated, closest to $z = 0$. The line of sight probes both near and far edges of the shell, giving rise to two mirror-image pairs of peaks. For a line of sight closer to r_s , however, v_r is closer to v_s , so the two velocity values at a given radius are close together and each bifurcated pair of peaks will merge into a single one. The distance from the shell edge where this merging happens depends on the shell thickness, but also on fiber size and spectral resolution. These spectral line shapes are similar to those presented in Ebrova et al. (2012), but include the broadening and changes in peak position induced by the range of energies of the stars in the shell. Their velocity profiles can be obtained from our expression by taking the limit $\delta_r \rightarrow 0$, or equivalently using Equation (17) for the distribution function instead of Equation (18).

The locations of the centers of the outer peaks are given by $\pm v_{\max}(R)$ as in Equation (23), but since the velocity profile convolves the physical density of the material with its shape in (r, v_r) space, we should replace r_s in the equation with r_{\max} , the radius of peak density defined in Equation (3). Thanks to the stream thickness these two radii are slightly offset from one another. The outer peak locations computed this way are marked in the left-hand panel of Figure 16; the colors match the respective lines of sight marked in the center panel. The interior peaks' locations can likewise be

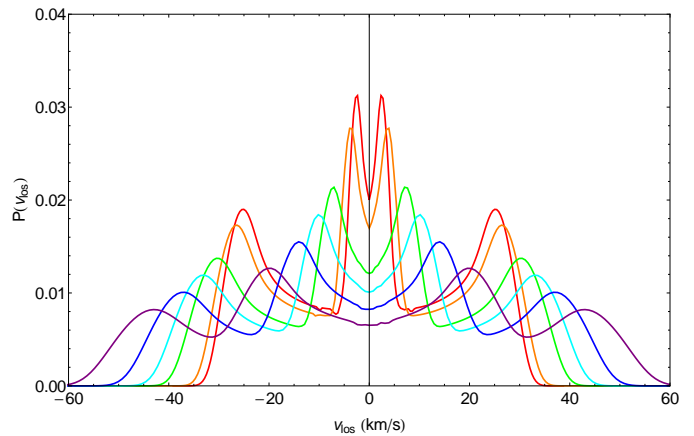


Figure 18. Changing the value of $g(r_s)$, for an otherwise constant set of shell parameters and constant line of sight (along the symmetry axis of the shell near its edge), alters the velocity profile. The colors from red to purple are the values $g(r_s) = \{1.2, 1.5, 2.5, 3.4, 4.9, 7.6\} \text{ km s}^{-1} \text{ Myr}^{-1}$, corresponding to the gravitational force at $r = \{90, 80, 60, 50, 40, 30\} \text{ kpc}$ in the potential of simulations A-C and the range $r = 10 - 35 \text{ kpc}$ in the potential of simulation D. Stronger gravity results in a broader profile and lower peaks. The shell is the same model as in Figure 16; the orange profiles in the two figures correspond.

computed by letting $v_s \rightarrow -v_s$ in Equation (23) and again using $r_s \rightarrow r_{\max}$. So one way to measure g_s from spectra is similar to that discussed in Section 6 for discrete measurements, except that since the spectra probe the integrated light they will automatically measure the correct v_{\max} . However, the caveats about potential flattening and angular span discussed above still apply: for this reason it is important to understand the geometry of the shell by fitting the surface brightness profile in order to judge whether Equation (23) is applicable.

The inclination and angular span of the shell have an effect on the integrated velocity profile just as they do on the distribution in (R, v_z) . Considering the angular cutoffs in Figure 7, we see that the limited angular span of visible material restricts the velocity profile so that only one of the two pairs of peaks (one of the two intersections with r_s) is probed. For θ_s near $\pi/2$, the line of sight is roughly symmetric around $z = 0$ so the two pairs of peaks are both the same height, but for inclinations much different from this, lines of sight can intersect only some of the peaks and cut off others, as shown in Figure 17. Knowing the geometry of the shell beforehand from image fitting is therefore a great advantage when deciding where to place fibers, since the chosen lines of sight must intersect all the peaks in order to measure $v_{z,\max}$.

The profile in even a single fiber is sensitive to $g(r_s)$ (Figure 18), so g_s can in principle be recovered by fitting the full velocity profiles of even a handful of lines of sight, even if there are too few to get the variation of the peak widths and fit Equation (23). Fitting the lineshape also allows the combination of multiple fibers (for example, from an integral-field unit) to improve the data quality. We will assess the quantitative ability of this strategy to recover g_s in future work.

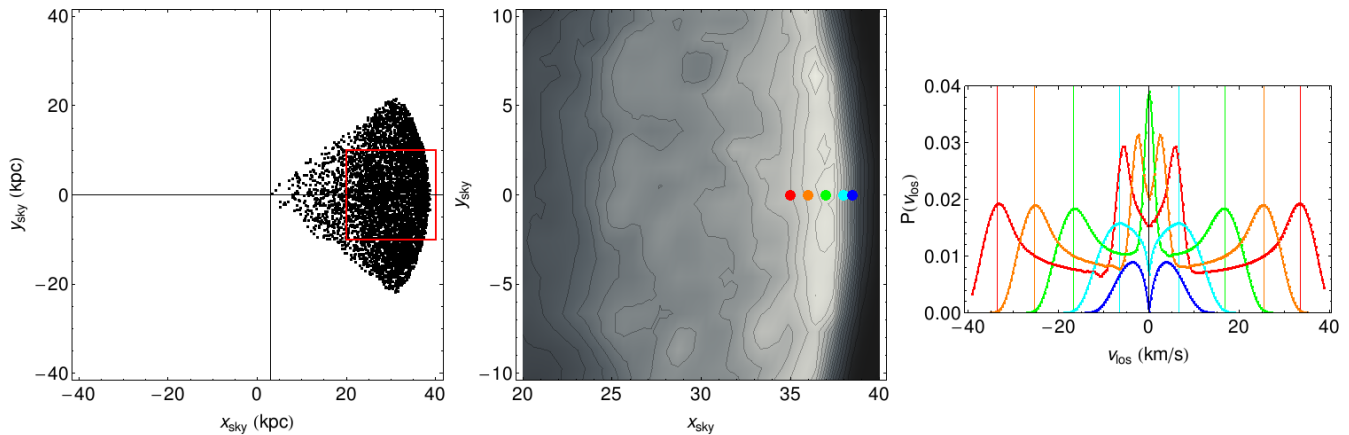


Figure 16. Equation (32) can be implemented numerically to predict the velocity distribution for different lines of sight through a shell. Here we illustrate using a particle realization (left) of a shell modeled on the one shown in green in Figure 11. The different lines of sight (center; colored circles) are overlaid on a surface-brightness map of the model convolved with a Gaussian of $2r_{\text{fib}}$; the red box in the right-hand panel indicates the area shown in the center panel. This example uses a fiber radius of 0.5 kpc, equivalent to 4 arcsec at 25 Mpc, shown to scale in the colored circles of the center panel. The different lines of sight produce different velocity profiles (right) based on their distance from the shell edge. Farther from the edge the profile has two pairs of peaks, while close to the shell edge only one pair of peaks is visible. The colored vertical lines mark the outer peak locations predicted by Equation (23) as described in the text.

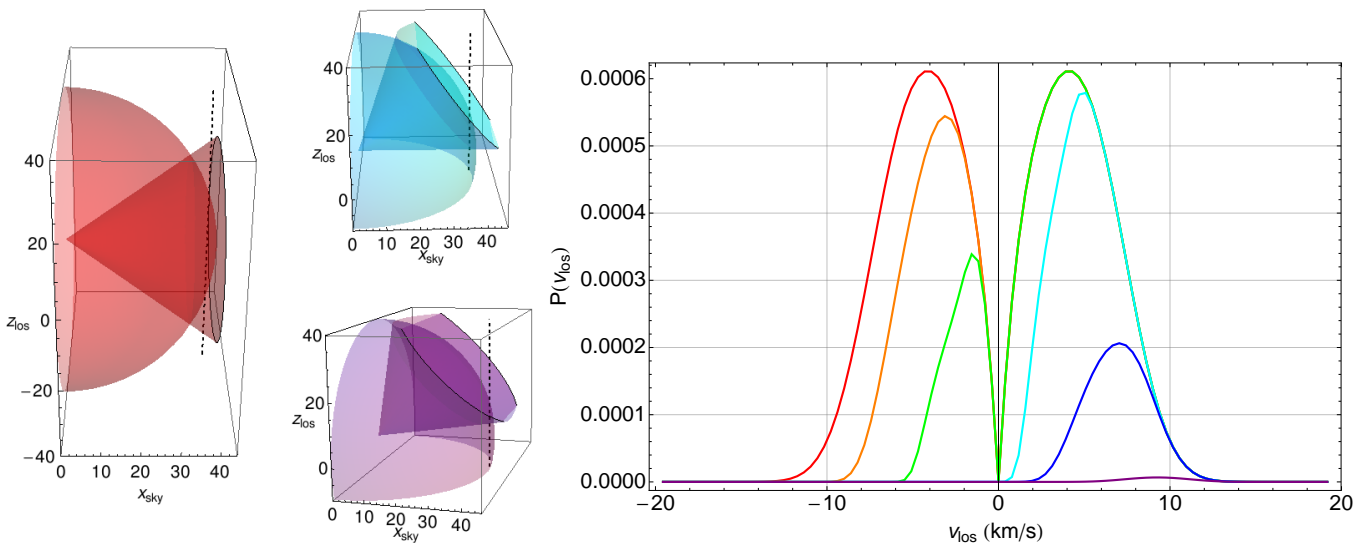


Figure 17. The precise angle of the shell with respect to the line of sight influences the velocity profile. In this example, as the shell is inclined more and more away from the observer the line of sight (black dashed line in the left panel) starts to probe a smaller range of radii. In this panel the shell occupies the intersection volume between the cone and the sphere. The angular cutoff causes an alteration in the velocity profile as a smaller range of velocities is included (right). The right panel shows the integrated velocity profile along the same line of sight for the inclination angles $\theta_s = \{90, 63, 60, 53, 49, 45\}^\circ$ for red through purple; the colors in the two panels correspond with each other and with the colors in Figure 12.

8 DISCUSSION AND CONCLUSIONS

In this paper we have presented a simple analytical model that successfully describes tidal shells with a handful of parameters, some of which are linked to the masses of the two interacting galaxies. The model includes self-consistent expressions for the surface brightness (Sections 3 and 5) and phase-space distribution (Section 4) of material near the edge of a single tidal shell. Given an image of a shell, the surface-brightness model jointly constrains the fine-grained phase-space density of the satellite material, f_0 , and the gravitational force at the shell edge through the parame-

ter κ , which is related to g_s as discussed in Section 3. The gravitational force $g_s(r_s)$ can be independently measured if line-of-sight velocities are also obtained. These can be either the velocities of individual point sources in a shell (Section 6) or velocity profiles obtained from integrated-light spectra (Section 7). If multiple shells are observed around a galaxy, each one gives an independent measurement of the gravitational force at the radius of its edge.

The model makes three assumptions about the two interacting galaxies: that the interaction orbit is radial, that the potential is spherical at the shell radius with negligible

tides, and that the satellite originally had a Maxwellian velocity distribution. We demonstrate in Section 3 that the assumptions of zero angular momentum, zero tides, and spherical symmetry in the potential are sufficient for caustics that appear as fan-like shells on the sky. The model is self-consistent in its treatment of the effect of the spread of orbital energies in a given shell, except for the assumption that the shell radius is constant. The model does not allow for a non-uniform distribution of material in the stream. In Section 4 we show that variations in the bulk velocity profile from the slightly nonuniform density along simulated streams are minor.

From the analytical model, we calculate different observable quantities—images, line-of-sight velocities, and velocity profiles—and demonstrate how each observable is sensitive to the radial component of the gravitational force, g_s , acting on the debris at the caustic. Thanks to the relatively simple forms of the phase space and spatial density, the calculations can be easily tailored to a specific observing paradigm. In Section 5 we show an example of how a simple geometric model, combined with the density profile, can be used to construct a basic but sufficient representation of simulated shells. Although images alone can only jointly constrain g_s and f_0 , the image fit is important to determine the geometry of the caustic, which is imprinted on line-of-sight velocity measurements.

In Section 6 we show that the maximum-velocity envelope of pointwise line-of-sight velocity measurements can be used to recover g_s to within better than 50 percent in most non-pathological cases; the crucial ingredient is to allow for the nonzero expansion velocity at the shell, which is a reflection of the energy spread of material in the caustic. The kinematic model fails when the potential is so flattened that it produces caustics in more than one dimension nearly simultaneously; this can be diagnosed by examining the symmetry and substructure of the velocity distribution around the mean v_z . The viewing angle of a shell can also produce an asymmetric velocity distribution, but only in a specific way; it cannot mimic the effect of a significantly flattened potential. The peaks of the velocity profiles presented in Section 7 depend on g_s in the same way as the velocity envelope, with the added advantage of probing the entire distribution along a given line of sight rather than sampling individual points. The value of g_s can also affect individual lineshapes, which could lead to another way of measuring this quantity.

One possible caveat to this work is that even for high-mass-ratio mergers like those studied here, the use of a static potential is not necessarily justified during the disruption of the satellite at pericenter, where the enclosed mass of the host galaxy can be comparable to the satellite mass. We conducted a high-resolution simulation using a live mass distribution, with conditions similar to simulation B, to study the effect of the response of the inner host galaxy. In agreement with previous work (e.g., Séguin et al. 1996), adding this response results in slightly thicker, more massive caustics at slightly smaller radii, but does not destroy the shells’ phase space structure. In addition, Mori & Rich (2008) performed Simulation D with a live representation of M31’s potential and found that the same shells-and-stream system formed in that case. This suggests that shells are not fundamentally

different in live host galaxies and that our results should extend to these cases.

Finally, each shell observed around a galaxy provides its own mass estimate at its own radius, without having to assume anything about the relationship between different shells. In galaxies with many shells, therefore, this technique could be used to obtain a rough dynamical mass profile of the host galaxy at otherwise inaccessible locations outside the luminous disk or ellipsoid. By using their distinctive morphology as a clue to their symmetry, shells will open for us a new window into the shapes and masses of external galaxies.

9 ACKNOWLEDGEMENTS

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APPENDIX A: POTENTIALS USED IN SIMULATIONS

Simulation A and Series B (see Table 1) use the spherical isochrone potential with total mass M and scale radius b described by

$$\Phi_{\text{iso}}^{\text{sph}}(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}, \quad (\text{A1})$$

where r is the spherical radial coordinate and G the gravitational constant.

Series C uses a flattened axisymmetric version of the same potential,

$$\Phi_{\text{iso}}^{\text{axi}}(R, z) = -\frac{GM}{b + \sqrt{R^2 + z^2/q^2 + b^2}}, \quad (\text{A2})$$

where q is the flattening parameter. In the limit $q \rightarrow 1$ one recovers the spherical isochrone potential. For simulation C we take $q = 0.7$ which produces an extremely flattened distribution; contours of the potential and radial force are shown in Figure A.

Simulation D uses the combined potential described in Table 2 and the references therein.

APPENDIX B: DERIVATION OF THE NORMALIZED COLD PHASE SPACE DENSITY

Near a caustic, the phase-space distribution function (DF) can be approximated by a line, a quadratic equation relating the galactocentric radius r and the radial velocity v_r :

$$f(r, v_r) = \mathcal{F}_0 \delta[r_s - r - \kappa(v_r - v_s)^2]. \quad (\text{B1})$$

Here we have suppressed notation indicating that there is no variation in v_θ or v_ϕ (two trivial delta functions) and also notation limiting the angular extent of the shell to some total solid angle Ω_s of the sphere. We will deal with this second notion by adjusting the limits of integration in angle. To determine the normalization \mathcal{F}_0 we impose the normalization condition

$$\int_0^{r_s} r^2 dr \int_{\Omega_s} d\Omega \int_{-\infty}^{\infty} dv_r dv_\theta dv_\phi f(r, v_r) = 1 \quad (\text{B2})$$

The integrals over v_θ and v_ϕ cancel, and the integral over the solid angle spanned by the shell, Ω_s , simply contributes a prefactor. Thus this expression simplifies to

$$\mathcal{F}_0 \Omega_s \int_0^{r_s} r^2 dr \int_{-\infty}^{\infty} dv_r \delta[r_s - r - \kappa(v_r - v_s)^2] = 1. \quad (\text{B3})$$

We now change variables to the argument of the delta function,

$$u = r_s - r - \kappa(v_r - v_s)^2 \equiv \Delta_r - \kappa(v_r - v_s)^2, \quad (\text{B4})$$

to integrate over v_r . Note that the quantity $\Delta_r \equiv r_s - r$ is always positive since the caustic only includes material at $r < r_s$.

This change of variables has two solutions for v_r ,

$$v_{r,\pm} = v_s \pm \sqrt{\frac{\Delta_r - u}{\kappa}} \quad (\text{B5})$$

To understand how to do the integral we therefore make a preliminary auxiliary change to the variable $x = v_r - v_s$, so that the integrand is symmetric in x :

$$I_{v_r} = \int_{-\infty}^{\infty} dx \delta(\Delta_r - \kappa x^2). \quad (\text{B6})$$

In terms of u ,

$$x_{\pm} = \pm \sqrt{\frac{\Delta_r - u}{\kappa}} \quad \text{and} \quad u = \Delta_r - \kappa x^2 \quad (\text{B7})$$

Now we break the integral up into positive and negative x pieces, so that it is clear which branch of the solution to take on each piece:

$$I_{v_r} = \int_{-\infty}^0 dx \delta(\Delta_r - \kappa x^2) + \int_0^{\infty} dx \delta(\Delta_r - \kappa x^2) \quad (\text{B8})$$

We note from the change of variables that at $x = \pm\infty$, $u = -\infty$, while at $x = 0$, u takes on its maximum (positive) value of Δ_r . So the zero point will still be included in both pieces of the integral, allowing the delta functions to be evaluated. To complete the change of variables we need the Jacobian dx/du :

$$\frac{dx_{\pm}}{du} = \mp \frac{1}{2\sqrt{\kappa(\Delta_r - u)}} \quad (\text{B9})$$

so that

$$\begin{aligned} I_{v_r} &= \int_{-\infty}^{\Delta_r} \frac{du}{2\sqrt{\kappa(\Delta_r - u)}} \delta(u) - \int_{\Delta_r}^{-\infty} \frac{du}{2\sqrt{\kappa(\Delta_r - u)}} \delta(u) \\ &= \int_{-\infty}^{\Delta_r} \frac{du}{\sqrt{\kappa(\Delta_r - u)}} \delta(u). \end{aligned} \quad (\text{B10})$$

Now we can evaluate the integral directly thanks to the delta function:

$$I_{v_r} = \frac{1}{\sqrt{\kappa(r_s - r)}}. \quad (\text{B11})$$

Note that this integral gives the functional form of the density distribution, which has the characteristic $1/\sqrt{r}$ falloff behind the caustic. This shows that we have chosen an appropriate form for the DF, since it gives the right density distribution when integrated over velocity.

Replacing the evaluated integral in the normalization condition gives

$$\mathcal{F}_0 \Omega_s \int_0^{r_s} \frac{r^2 dr}{\sqrt{\kappa(r_s - r)}} = 1. \quad (\text{B12})$$

The remaining integral can be evaluated quite easily to give the definition of \mathcal{F}_0 :

$$\frac{16\mathcal{F}_0 \Omega_s r_s^{5/2}}{15\sqrt{\kappa}} = 1, \quad \text{or} \quad \mathcal{F}_0 = \frac{15\sqrt{\kappa}}{16r_s^{5/2} \Omega_s}. \quad (\text{B13})$$

Thus, the complete expression for the DF is

$$f(r, v_r) = \frac{15\sqrt{\kappa}}{16r_s^{5/2} \Omega_s} \delta[r_s - r - \kappa(v_r - v_s)^2], \quad (\text{B14})$$

where we have suppressed the trivial delta functions on v_θ and v_ϕ .

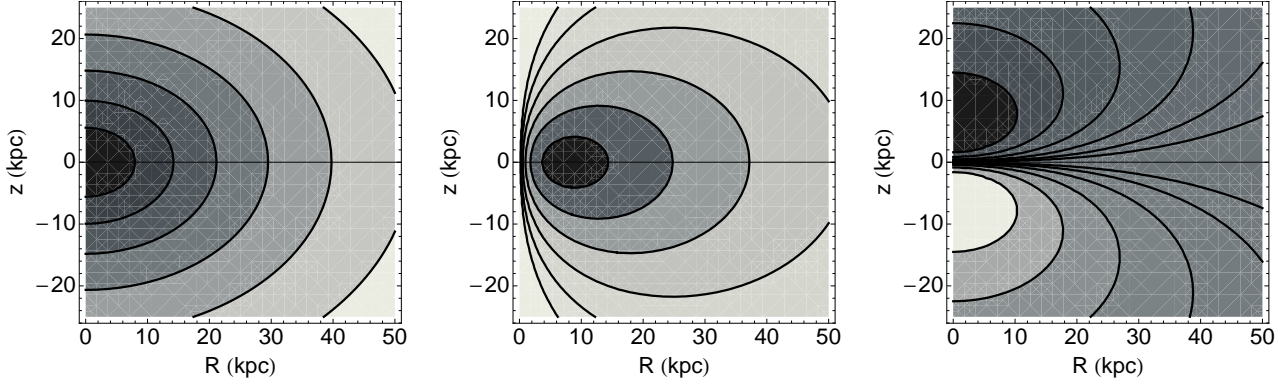


Figure A1. Contours of the potential (left), R force (center), and z force (right) for the flattened isochrone potential used in this work. In the left panel, contour spacing is 0.1 dex; in the center and right panels it is 0.25 dex.

APPENDIX C: DERIVATION OF THE PHASE-SPACE DENSITY; VERIFICATION OF THE DENSITY DISTRIBUTION

The phase-space density is obtained by averaging over an ensemble of cold distributions, each with a slightly different caustic radius:

$$f(r, v_r) \propto \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\delta_r^2}\right) \delta[r_s + x - r - \kappa(v_r - v_s)^2] x \equiv \mathcal{I}. \quad (\text{C1})$$

To perform the integral we change variables to the quantity inside the delta function,

$$u = r_s + x - r - \kappa(v_r - v_s)^2, \quad (\text{C2})$$

which has the same endpoints and a Jacobian of 1 ($du = dx$). Then the integral becomes trivial:

$$\mathcal{I} = \int_{-\infty}^{\infty} \exp\left\{-\frac{[u - r_s + r + \kappa(v_r - v_s)^2]^2}{2\delta_r^2}\right\} \delta(u) du, \quad (\text{C3})$$

so that

$$\mathcal{I} = \exp\left\{-\frac{[r_s - r - \kappa(v_r - v_s)^2]^2}{2\delta_r^2}\right\}. \quad (\text{C4})$$

We can verify that this expression for the phase density gives the same functional form as Equation (1) for the density ρ by integrating over v_r , although the normalization will be different since the phase density is normalized to 1. The density will be proportional to the integral of \mathcal{I} over v_r :

$$\rho \propto \int_{-\infty}^{\infty} \exp\left\{-\frac{[r_s - r - \kappa(v_r - v_s)^2]^2}{2\delta_r^2}\right\} dv_r. \quad (\text{C5})$$

We change variables to make the exponent Gaussian in the variable of integration,

$$u = \frac{r_s - r - \kappa(v_r - v_s)^2}{\sqrt{2}\delta_r}, \quad (\text{C6})$$

and determine the Jacobian:

$$dv_r = \mp \frac{\delta_r}{\sqrt{2\kappa(r_s - r - \sqrt{2}\delta_r u)}}, \quad (\text{C7})$$

where the upper sign is for the part of the integral where $v_r > v_s$ (the upper half of the parabola) and the lower sign is for the part when $v_r < v_s$ (the lower half). This implies that

we must break the integral into two parts with the following endpoints in u , taking into account the sign of the Jacobian:

$$\begin{aligned} -\infty < v_r < v_s &\rightarrow -\infty < u < \frac{r_s - r}{\sqrt{2}\delta_r} \\ v_s \leq v_r < \infty &\rightarrow \frac{r_s - r}{\sqrt{2}\delta_r} \leq u < \infty \end{aligned}$$

So the two parts of the integral are identical; thus we can say

$$\rho \propto \int_{-\infty}^{(r_s - r)/\sqrt{2}\delta_r} \frac{e^{-u^2} du}{\sqrt{r_s - r - \sqrt{2}\delta_r u}}. \quad (\text{C8})$$

This integral can be evaluated as a combination of Bessel functions depending on the sign of $r_s - r$. Behind the caustic, this quantity is positive; in front it is negative, so we get a piecewise solution with the same domains as Equation (1). Performing the integral gives the result:

$$\rho \propto \sqrt{|r_s - r|} e^{-\frac{(r_s - r)^2}{4\delta_r^2}} \mathcal{B}\left[\frac{(r_s - r)^2}{4\delta_r^2}\right], \quad (\text{C9})$$

which is the same functional form as Equation (1).

APPENDIX D: EQUATIONS FOR A ROTATED CONE

A cone of height h and opening angle α , with its point at the origin and its axis of symmetry along the z axis has the parametric equations (in Cartesian coordinates)

$$x = (h - u) \cos \vartheta \tan \alpha \quad (\text{D1})$$

$$y = (h - u) \sin \vartheta \tan \alpha \quad (\text{D2})$$

$$z = h - u \quad (\text{D3})$$

where the parameter $u \in [0, h]$ describes the distance from the base of the cone and the parameter $\vartheta \in [0, 2\pi]$ describes the azimuthal location on the cone. If the cone is rotated so that its axis of symmetry points along the unit vector

$$\hat{n} = \sin \theta_s \cos \phi_s \hat{x} + \sin \theta_s \sin \phi_s \hat{y} + \cos \theta_s \hat{z} \quad (\text{D4})$$

then the equations become

$$x = (h - u) \{ \cos \vartheta [1 - \cos^2 \phi_s (1 - \cos \theta_s)] \tan \alpha - \sin \vartheta [\cos \phi_s \sin \phi_s \tan \alpha (1 - \cos \theta_s)] + \sin \theta_s \cos \phi_s \} \quad (\text{D5})$$

$$y = (h - u) \{ -\cos \vartheta [\cos \phi_s \sin \phi_s \tan \alpha (1 - \cos \theta_s)] + \sin \vartheta [1 - \sin^2 \phi_s (1 - \cos \theta_s)] \tan \alpha + \sin \theta_s \cos \phi_s \} \quad (\text{D6})$$

$$z = (h - u) [-\cos(\vartheta - \phi_s) \sin \theta_s \tan \alpha + \cos \theta_s] \quad (\text{D7})$$

To determine z_{\min} and z_{\max} as a function of x and y , we solve the system of the x and y parametric equations to obtain the parameters u and ϑ . The system can have zero, one, or two solutions depending upon the values of x and y : zero solutions for lines of sight that do not intersect the cone, one solution for lines of sight that intersect the cone once, and two solutions for lines of sight that intersect the cone twice. In cases where there is one solution, the other limit can safely be taken to be $\pm\infty$, where the plus sign is taken if the bulk of the cone is in front of the intersection point (the solution is the lower bound of z), and the minus sign is taken if the cone is behind the intersection (the solution is the upper bound of z). The system can always be solved analytically and we present the solution here. First we will define some auxiliary quantities to make the notation simpler. We extract the θ_s , ϕ_s , and α dependence of Equations (D5-D6) into coefficients that need only be calculated once for a given cone:

$$A_x = \tan \alpha [1 - \cos^2 \phi_s (1 - \cos \theta_s)] \quad (\text{D8})$$

$$B_x = \tan \alpha \cos \phi_s \sin \phi_s (1 - \cos \theta_s) \quad (\text{D9})$$

$$C_x = \sin \theta_s \cos \phi_s \quad (\text{D10})$$

$$A_y = B_x \quad (\text{D11})$$

$$B_y = \tan \alpha [1 - \sin^2 \phi_s (1 - \cos \theta_s)] \quad (\text{D12})$$

$$C_y = \sin \theta_s \sin \phi_s, \quad (\text{D13})$$

so that Equations (D5-D6) become

$$x = (h - u) (A_x \cos \vartheta - B_x \sin \vartheta + C_x) \quad (\text{D14})$$

$$y = (h - u) (-A_y \cos \vartheta + B_y \sin \vartheta + C_y). \quad (\text{D15})$$

We can solve for ϑ by dividing the two equations, noting that if $x = 0$ or $y = 0$ then we are at the point of the cone ($u = h$) and ϑ is degenerate. We obtain an equation for ϑ in terms of the ratio $\eta \equiv x/y$:

$$\mathcal{S} \sin \vartheta - \mathcal{C} \cos \vartheta + \mathcal{K} = 0, \quad (\text{D16})$$

where

$$\mathcal{C} \equiv A_x + \eta A_y \quad (\text{D17})$$

$$\mathcal{S} \equiv B_x + \eta B_y \quad (\text{D18})$$

$$\mathcal{K} \equiv -C_x + \eta C_y. \quad (\text{D19})$$

Equation (D16) can be recast as a quadratic equation for either $\sin \vartheta$ or $\cos \vartheta$. To avoid using inverse trigonometric functions (and the associated difficulties in choosing the right branch) we simply solve for both and use them in the rest of the solution:

$$(\cos \vartheta)_{\pm} = \frac{\mathcal{K}\mathcal{C} \pm \mathcal{S}\mathcal{D}}{\mathcal{C}^2 + \mathcal{S}^2} \quad (\text{D20})$$

$$(\sin \vartheta)_{\pm} = \frac{\mathcal{K}\mathcal{S} \pm \mathcal{C}\mathcal{D}}{\mathcal{C}^2 + \mathcal{S}^2} \quad (\text{D21})$$

where \mathcal{D}^2 is the discriminant

$$\mathcal{D}^2 \equiv \mathcal{C}^2 + \mathcal{S}^2 - \mathcal{K}^2. \quad (\text{D22})$$

As usual, if $\mathcal{D}^2 < 0$ the point is outside the cone and the equation has no real roots; otherwise it has two real roots. When two solutions exist for ϑ , one or both of them may lead to a value of u outside its allowed range.

Having determined ϑ either Equation (D5) or Equation (D6) can be used to determine the quantity $h - u$ that is necessary to find the limits on z :

$$(h - u)_{\pm} = \frac{x}{A_x (\cos \vartheta)_{\pm} - B_x (\sin \vartheta)_{\pm} + C_x} = \frac{y}{-A_y (\cos \vartheta)_{\pm} + B_y (\sin \vartheta)_{\pm} + C_y} \quad (\text{D23})$$

The quantity $h - u$ should be in the range $(0, h]$ —if it is not, that value of ϑ is discarded as a root and one of the limits in z goes to $\pm\infty$.

Finally, the limits z_{\pm} are given by plugging in the valid solutions for u and ϑ :

$$z_{\pm} = (h - u)_{\pm} \{ \cos \theta_s - \sin \theta_s \tan \alpha [(\cos \vartheta)_{\pm} \cos \phi_s + (\sin \vartheta)_{\pm} \sin \phi_s] \} \quad (\text{D24})$$

By inspection, we see that z_{\min} is not always equal to z_- and z_{\max} is not always z_+ since both $(\cos \vartheta)_{\pm}$ and $(\sin \vartheta)_{\pm}$ can take any sign: the roots must be compared and the smaller assigned to z_{\min} . If one root is out of range in u , its out-of-range z value can still be calculated (for this work the height h of the cone is arbitrary because the density function effectively cuts off the cone along a spherical segment) and compared to the value of the in-range root to determine whether the out-of-range limit is z_{\min} or z_{\max} .

In constructing a fitting routine, the partial derivatives of Σ with respect to the projection parameters α, θ_s, ϕ_s and to the density profile parameters $r_s, \delta_r, \kappa, f_0$ may be needed. In the following, we define the shorthand

$$[\cos(\vartheta - \phi_s)]_{\max} \equiv (\cos \vartheta)_{\max} \cos \phi_s + (\sin \vartheta)_{\max} \sin \phi_s$$

$$[\sin(\vartheta - \phi_s)]_{\max} \equiv (\sin \vartheta)_{\max} \cos \phi_s - (\cos \vartheta)_{\max} \sin \phi_s$$

where $(\cos \vartheta)_{\max}$ and $(\sin \vartheta)_{\max}$ are taken to be the roots that lead to the value of z_{\max} , and likewise

$$[\cos(\vartheta - \phi_s)]_{\min} \equiv (\cos \vartheta)_{\min} \cos \phi_s + (\sin \vartheta)_{\min} \sin \phi_s$$

$$[\sin(\vartheta - \phi_s)]_{\min} \equiv (\sin \vartheta)_{\min} \cos \phi_s - (\cos \vartheta)_{\min} \sin \phi_s$$

for the roots leading to the value of z_{\min} . A similar notation is used to denote the appropriate root of $(h - u)$. We also use the shorthand

$$\rho(z_{\max}) \equiv \rho(\sqrt{x^2 + y^2 + z_{\max}^2}) \quad (\text{D25})$$

in the following, since x and y are understood to be constant. With these definitions, the derivatives with respect to the

projection parameters are:

$$\frac{\partial \Sigma}{\partial \alpha} = \frac{\sin \theta_s \sec^2 \alpha}{\Upsilon} \left\{ \rho(z_{\min}) [\cos(\vartheta - \phi_s)]_{\min} (h - u)_{\min} - \rho(z_{\max}) [\cos(\vartheta - \phi_s)]_{\max} (h - u)_{\max} \right\} \quad (\text{D26})$$

$$\begin{aligned} \frac{\partial \Sigma}{\partial \theta_s} = \frac{1}{\Upsilon} \left\{ \rho(z_{\min}) [(h - u)_{\min} \sin \theta_s + [\cos(\vartheta - \phi_s)]_{\min} \cos \theta_s \tan \alpha] \right. \\ \left. - \rho(z_{\max}) [(h - u)_{\max} \sin \theta_s + [\cos(\vartheta - \phi_s)]_{\max} \cos \theta_s \tan \alpha] \right\} \quad (\text{D27}) \end{aligned}$$

$$\frac{\partial \Sigma}{\partial \phi_s} = \frac{\sin \theta_s \tan \alpha}{\Upsilon} \left\{ \rho(z_{\min}) (h - u)_{\min} [\sin(\vartheta - \phi_s)]_{\min} - \rho(z_{\max}) (h - u)_{\max} [\sin(\vartheta - \phi_s)]_{\max} \right\} \quad (\text{D28})$$

The derivatives with respect to the profile parameters are all integrals of the derivatives of the density, of the form

$$\frac{\partial \Sigma}{\partial \pi_i} = \frac{1}{\Upsilon} \int_{z_{\min}}^{z_{\max}} \frac{\partial \rho}{\partial \pi_i} dz \quad (\text{D29})$$

for a given parameter π_i , since the limits of the integral and the variable being integrated over do not depend on any of the π_i and the function ρ , although defined as a piecewise function in Equation (1), is continuous over the entire integration range. To compactly write derivatives of ρ with respect to the parameters, we expand the definition of \mathcal{B} to include other Bessel functions:

$$\mathcal{B}_n(u) \equiv \begin{cases} \frac{\pi}{2} [\mathcal{I}_{(2n+1)/4}(u) + \mathcal{I}_{-(2n+1)/4}(u)] & r \leq r_s \\ (-1)^{n+1} \frac{\pi}{2} [\mathcal{I}_{(2n+1)/4}(u) - \mathcal{I}_{-(2n+1)/4}(u)] & r > r_s \end{cases} \quad (\text{D30})$$

with $u \equiv (r - r_s)^2 / 4\delta_r^2$ as before. The definition used in Equation (1) is equivalent to \mathcal{B}_0 in this new notation. With this simplification, the derivatives $\partial \rho / \partial \pi_i$ are:

$$\begin{aligned} \frac{\partial \rho}{\partial r_s} = \frac{f_0}{\sqrt{2\pi\kappa}} e^{-u} \frac{\sqrt{|r - r_s|}}{r - r_s} \times \\ \times \left\{ u [2\mathcal{B}_0(u) - \mathcal{B}_2(u) - \mathcal{B}_1(u)] - \frac{1}{2}\mathcal{B}_0(u) \right\} \quad (\text{D31}) \end{aligned}$$

$$\begin{aligned} \frac{\partial \rho}{\partial \delta_r} = \frac{f_0}{\sqrt{2\pi\kappa}} e^{-u} \sqrt{|r - r_s|} \frac{u}{\delta_r} \times \\ \times [2\mathcal{B}_0(u) - \mathcal{B}_2(u) - \mathcal{B}_1(u)] \quad (\text{D32}) \end{aligned}$$

$$\frac{\partial \rho}{\partial f_0} = \frac{\rho}{f_0} \quad (\text{D33})$$

$$\frac{\partial \rho}{\partial \kappa} = -\frac{\rho}{2\kappa} \quad (\text{D34})$$